SET- 36

Radiation

① $\vec{B} = \vec{\nabla} \times \vec{A}$

$$\vec{E} = -\vec{\nabla}\phi - \frac{\partial \vec{A}}{\partial t}$$

$$\left(-\nabla^2 + \frac{1}{c^2}\frac{\partial^2}{\partial t^2}\right)\phi(\vec{r},t) = \frac{1}{\epsilon_0}\rho(\vec{r},t)$$

$$\left(-\nabla^2 + \frac{1}{c^2}\frac{\partial^2}{\partial t^2}\right)\vec{A}(\vec{r},t) = \mu_0 \vec{J}(\vec{r},t)$$

② We introduce the Green's function

$$\left(-\nabla^2 + \frac{1}{c^2}\frac{\partial^2}{\partial t^2}\right)\frac{1}{4\pi}G(\vec{r}-\vec{r}', t-t') = \delta^{(3)}(\vec{r}-\vec{r}')\,\delta(t-t')$$

which has solution

$$G(\vec{r}-\vec{r}', t-t') = \frac{1}{|\vec{r}-\vec{r}'|}\,\delta\left(t-t'-\frac{1}{c}|\vec{r}-\vec{r}'|\right)$$

③ Using ② in ① we have the potentials

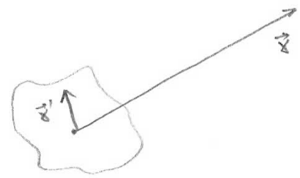
$$\phi(\vec{r},t) = \frac{1}{4\pi\epsilon_0}\int d^3r' \int_{-\infty}^{+\infty} dt' \, \frac{\rho(\vec{r}',t')}{|\vec{r}-\vec{r}'|}\,\delta\left(t-t'-\frac{1}{c}|\vec{r}-\vec{r}'|\right)$$

$$= \frac{1}{4\pi\epsilon_0}\int d^3r' \, \frac{\rho\left(\vec{r}', t-\frac{1}{c}|\vec{r}-\vec{r}'|\right)}{|\vec{r}-\vec{r}'|}$$

$$\vec{A}(\vec{r},t) = \frac{\mu_0}{4\pi}\int d^3r' \int_{-\infty}^{+\infty} dt' \, \frac{\vec{J}(\vec{r}',t')}{|\vec{r}-\vec{r}'|}\,\delta\left(t-t'-\frac{1}{c}|\vec{r}-\vec{r}'|\right)$$

$$= \frac{\mu_0}{4\pi}\int d^3r' \, \frac{\vec{J}\left(\vec{r}', t-\frac{1}{c}|\vec{r}-\vec{r}'|\right)}{|\vec{r}-\vec{r}'|}$$

4



$\tau \rightarrow$ characteristic time involved in the process.

(i) Statics : $|\vec{r}-\vec{r}'| << c\tau$

(ii) Radiation : $|\vec{r}-\vec{r}'| >> c\tau$ $\quad (|\vec{r}| >> |\vec{r}'|)$

5 Case (i) corresponds to

$$t - \frac{1}{c}|\vec{r}-\vec{r}'| \approx t$$

which when used in 3 leads to

$$\phi(\vec{r},t) \approx \frac{1}{4\pi\epsilon_0}\int d^3r' \, \frac{\rho(\vec{r}',t)}{|\vec{r}-\vec{r}'|},$$

$$\vec{A}(\vec{r},t) \approx \frac{\mu_0}{4\pi}\int d^3r' \, \frac{\vec{J}(\vec{r}',t)}{|\vec{r}-\vec{r}'|},$$

which is the case of statics.

6 We shall concentrate on radiation. In the radiation approximation we have.

$$\hat{r} = \frac{\vec{r}}{r}$$

$$|\vec{r}-\vec{r}'| = \sqrt{r^2 + r'^2 - 2\vec{r}\cdot\vec{r}'}$$

$$= r\sqrt{1 + \left(\frac{r'}{r}\right)^2 - 2\frac{\vec{r}\cdot\vec{r}'}{r^2}}$$

$$= r\sqrt{1 - 2\hat{r}\cdot\frac{\vec{r}'}{r} + O\left(\frac{r'}{r}\right)^2}$$

$$\sqrt{1-x} = 1 - \frac{1}{2}x + \dots$$

$$= r\left[1 - \hat{r}\cdot\frac{\vec{r}'}{r} + O\left(\frac{r'}{r}\right)^2\right]$$

$$O^2 = O\left(\frac{r'}{r}\right)^2.$$

$$= r - \hat{r}\cdot\vec{r}' + r\,O^2$$

$$\frac{1}{|\vec{r}-\vec{r}'|} = \frac{1}{r\left[1 - \hat{r}\cdot\frac{\vec{r}'}{r} + O^2\right]}$$

$$\frac{1}{1-x} = 1 + x + \dots$$

$$= \frac{1}{r}\left[1 + O'\right]$$

7 Using 6 in 3 we have the potentials in the radiation approximation to be.

$$\phi(\vec{r},t) = \frac{1}{4\pi\epsilon_0}\frac{1}{r}\int d^3r'\,\rho\left(\vec{r}', t - \frac{r}{c} + \frac{\hat{r}\cdot\vec{r}'}{c}\right) + O^2$$

$$\vec{A}(\vec{r},t) = \frac{\mu_0}{4\pi}\frac{1}{r}\int d^3r'\,\vec{J}\left(\vec{r}', t - \frac{r}{c} + \frac{\hat{r}\cdot\vec{r}'}{c}\right) + O^2$$

$$t_r = t - \frac{r}{c} + \frac{\hat{r}\cdot\vec{r}'}{c}$$

8 The fields are obtained from 7 using

$$\vec{B} = \vec{\nabla} \times \vec{A}$$

$$\vec{E} = -\vec{\nabla}\phi - \frac{\partial \vec{A}}{\partial t}.$$

9 Let us evaluate

$$\vec{\nabla}\phi(\vec{r},t) = \vec{\nabla}\,\frac{1}{4\pi\epsilon_0}\,\frac{1}{r}\int d^3r'\,\rho\Big(\vec{r}',\,\overbrace{t - \frac{r}{c} + \frac{\hat{r}\cdot\vec{r}'}{c}}^{t_r}\Big)$$

$$= \frac{1}{4\pi\epsilon_0}\left(\vec{\nabla}\frac{1}{r}\right)\int d^3r'\,\rho(\vec{r}',t_r) + \frac{1}{4\pi\epsilon_0}\,\frac{1}{r}\int d^3r'\,(\vec{\nabla}t_r)\left\{\frac{\partial}{\partial t'}\rho(\vec{r}',t')\right\}_{t'=t_r}$$

10 $$\vec{\nabla}\frac{1}{r} = -\frac{\hat{r}}{r^2}$$

$$\vec{\nabla}t_r = \vec{\nabla}\left(t - \frac{r}{c} + \frac{\hat{r}\cdot\vec{r}'}{c}\right)$$

$$= -\frac{1}{c}\vec{\nabla}r + \frac{1}{c}(\vec{\nabla}\hat{r})\cdot\vec{r}'$$

$$= -\frac{\hat{r}}{c} + \frac{1}{c}\frac{1}{r}\left(\overleftrightarrow{I} - \hat{r}\hat{r}\right)\cdot\vec{r}'$$

$$= -\frac{\hat{r}}{c} + \frac{1}{c}\frac{1}{r}\left(\vec{r}' - \hat{r}\,\hat{r}\cdot\vec{r}'\right)$$

$$= -\frac{\hat{r}}{c} - \frac{1}{c}\,\hat{r}\times\left(\hat{r}\times\frac{\vec{r}'}{r}\right)$$

$$\vec{\nabla}\hat{r} = \vec{\nabla}\left(\frac{\vec{r}}{r}\right)$$

$$= \frac{(\vec{\nabla}\vec{r})}{r} + \vec{r}\left(\vec{\nabla}\frac{1}{r}\right)$$

$$= \frac{\overleftrightarrow{I}}{r} - \frac{\vec{r}\,\hat{r}}{r^2}.$$

$$= \frac{1}{r}\left(\overleftrightarrow{I} - \hat{r}\hat{r}\right)$$

(11) Using (10) in (9)

$$\vec{\nabla}\phi(\vec{r},t) = \frac{1}{4\pi\epsilon_0}\,\frac{\hat{r}}{r^2}\int d^3r'\,\rho(\vec{r}',t_r) \qquad \nearrow \frac{1}{r^2}$$

$$- \frac{1}{4\pi\epsilon_0}\,\frac{\hat{r}}{r}\,\frac{1}{c}\int d^3r'\left\{\frac{\partial}{\partial t'}\rho(\vec{r}',t')\right\}_{t'=t_r} \qquad \nearrow \frac{1}{r}$$

$$- \frac{1}{4\pi\epsilon_0}\,\frac{1}{r}\,\frac{1}{c}\int d^3r'\;\hat{r}\times\left(\hat{r}\times\frac{\vec{r}'}{r}\right)\left\{\frac{\partial}{\partial t'}\rho(\vec{r}',t')\right\}_{t'=t_r} \qquad \nearrow \frac{1}{r^2}$$

$$= -\frac{1}{c}\,\frac{1}{4\pi\epsilon_0}\,\frac{\hat{r}}{r}\int d^3r'\left\{\frac{\partial}{\partial t'}\rho(\vec{r}',t')\right\}_{t'=t_r} + O^2$$

(12) Similarly we can derive.

$$\vec{\nabla}\times\vec{A}(\vec{r},t) = -\frac{1}{c}\,\frac{\mu_0}{4\pi}\,\frac{\hat{r}}{r}\times\int d^3r'\left\{\frac{\partial}{\partial t'}\vec{J}(\vec{r}',t')\right\}_{t'=t_r} + O^2.$$

$$\frac{\partial}{\partial t}\vec{A}(\vec{r},t) = \frac{\mu_0}{4\pi}\,\frac{1}{r}\int d^3r'\,\frac{\partial}{\partial t}\vec{J}\left(\vec{r}', t-\frac{r}{c}+\frac{\hat{r}\cdot\vec{r}'}{c}\right) + O^2$$

$$= \frac{\mu_0}{4\pi}\,\frac{1}{r}\int d^3r'\left\{\frac{\partial}{\partial t'}\vec{J}(\vec{r}',t')\right\}_{t'=t_r} + O^2.$$

(13) Using (11) and (12) in (8) we have.

$$c\vec{B}(\vec{r},t) = -\hat{r}\times\frac{\mu_0}{4\pi}\,\frac{1}{r}\int d^3r'\left\{\frac{\partial}{\partial t'}\vec{J}(\vec{r}',t')\right\}_{t'=t_r} + O^2.$$

$$\vec{E}(\vec{r},t) = +\frac{1}{c}\,\frac{1}{4\pi\epsilon_0}\,\frac{\hat{r}}{r}\int d^3r'\left\{\frac{\partial}{\partial t'}\rho(\vec{r}',t')\right\}_{t'=t_r} - \frac{\mu_0}{4\pi}\,\frac{1}{r}\int d^3r'\left\{\frac{\partial}{\partial t'}\vec{J}(\vec{r}',t')\right\}_{t'=t_r} + O^2$$

(14) Maxwell's equations implies the local charge conservation,

$$\frac{\partial}{\partial t'} \rho(\vec{r}', t') + \vec{\nabla}' \cdot \vec{J}(\vec{r}', t') = 0$$

using which we have.

$$\left\{\frac{\partial}{\partial t'} \rho(\vec{r}', t')\right\}_{t'=t_r} = - \left\{\vec{\nabla}' \cdot \vec{J}(\vec{r}', t')\right\}_{t'=t_r = t - \frac{r}{c} + \frac{\hat{r}\cdot\vec{r}'}{c}}$$

$$= - \vec{\nabla}' \cdot \vec{J}(\vec{r}', t_r) + (\vec{\nabla}' t_r) \cdot \left\{\frac{\partial}{\partial t'} \vec{J}(\vec{r}', t')\right\}_{t'=t_r}$$

$$= - \vec{\nabla}' \cdot \vec{J}(\vec{r}', t_r) + \frac{\hat{r}}{c} \cdot \left\{\frac{\partial}{\partial t'} \vec{J}(\vec{r}', t')\right\}_{t'=t_r} \qquad \left(\vec{\nabla}' t_r = \frac{\hat{r}}{c}\right)$$

(15) Using (14) we have.

$$\int d^3r' \left\{\frac{\partial}{\partial t'} \rho(\vec{r}', t')\right\}_{t'=t_r} = - \int d^3r' \, \vec{\nabla}' \cdot \vec{J}(\vec{r}', t_r) \longrightarrow \begin{array}{l} = 0 \text{ if} \\ \vec{J} \text{ is confined.} \end{array}$$

$$+ \frac{\hat{r}}{c} \cdot \int d^3r' \left\{\frac{\partial}{\partial t'} \vec{J}(\vec{r}', t')\right\}_{t'=t_r}$$

$$= \frac{\hat{r}}{c} \cdot \int d^3r' \left\{\frac{\partial}{\partial t'} \vec{J}(\vec{r}', t')\right\}_{t'=t_r}.$$

16 Using 15 in the expression for electric field in 13 we have.

$$\vec{E}(\vec{r},t) = \frac{1}{c}\frac{1}{4\pi\epsilon_0}\frac{\hat{r}}{r}\frac{\hat{r}}{c}\cdot\int d^3r'\left\{\frac{\partial}{\partial t'}\vec{J}(\vec{r}',t')\right\}_{t'=t_r} - \frac{\mu_0}{4\pi}\frac{1}{r}\int d^3r'\left\{\frac{\partial}{\partial t'}\vec{J}(\vec{r}',t')\right\}_{t'=t_r} + O^2$$

$$= \hat{r}\,\hat{r}\cdot\frac{\mu_0}{4\pi}\frac{1}{r}\int d^3r'\left\{\frac{\partial}{\partial t'}\vec{J}(\vec{r}',t')\right\}_{t'=t_r} - \frac{\mu_0}{4\pi}\frac{1}{r}\int d^3r'\left\{\frac{\partial}{\partial t'}\vec{J}(\vec{r}',t')\right\}_{t'=t_r} + O^2$$

$$= \hat{r}\times\left(\hat{r}\times\frac{\mu_0}{4\pi}\frac{1}{r}\int d^3r'\left\{\frac{\partial}{\partial t'}\vec{J}(\vec{r}',t')\right\}_{t'=t_r}\right) + O^2.$$

$$= -\,\hat{r}\times c\vec{B}(\vec{r},t) + O^2 \qquad \text{(using 13).}$$

17 Thus, we have.

$$c\vec{B}(\vec{r},t) = -\,\hat{r}\times\frac{\mu_0}{4\pi}\frac{1}{r}\int d^3r'\left\{\frac{\partial}{\partial t'}\vec{J}(\vec{r}',t')\right\}_{t'=t_r} + O^2.$$

$$\vec{E}(\vec{r},t) = -\,\hat{r}\times c\vec{B}(\vec{r},t) + O^2.$$

18 Note that

$$\hat{r}\times\vec{E} = -\,\hat{r}\times(\hat{r}\times c\vec{B})$$

$$= -\,\hat{r}\,\underbrace{(\hat{r}\cdot c\vec{B})}_{=0} + c\vec{B}$$

$$= c\vec{B}$$

Thus, $\hat{r}$, $\vec{E}$, and $\vec{B}$ are orthogonal to each other

(19) Radiation fields are characterized by

$$\left.\begin{aligned} c\vec{B} &= \hat{r} \times \vec{E} + O^2. \\ \vec{E} &= -\hat{r} \times c\vec{B} + O^2. \end{aligned}\right\} \quad \hat{r},\ \vec{E},\ \text{and}\ \vec{B}\ \text{are orthogonal}$$

$$c^2B^2 = E^2 + O^2. \quad \longrightarrow \text{electric energy} = \text{magnetic energy.}$$

These properties are also satisfied by plane waves.