SET-34 A

Review of Lorentz transformations

① A rotation about z-axis

$$\begin{pmatrix} x' \\ y' \end{pmatrix} = \begin{pmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix}$$

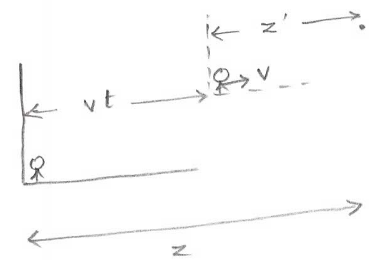
$$\begin{pmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{pmatrix} \begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$$

keeps the distance

$$x'^2 + y'^2 = x^2 + y^2.$$

unchanged.

② Boost in z-direction:



$$z' = (z - vt)\,\gamma$$

$$ct' = \left(-\frac{v}{c} z + ct\right)\gamma$$

$$\beta = \frac{v}{c} \qquad \gamma = \frac{1}{\sqrt{1-\beta^2}} \qquad L = \begin{pmatrix} \gamma & \beta\gamma \\ \beta\gamma & \gamma \end{pmatrix}$$

$$\begin{pmatrix} z' \\ ct' \end{pmatrix} = \begin{pmatrix} \gamma & -\beta\gamma \\ -\beta\gamma & \gamma \end{pmatrix} \begin{pmatrix} z \\ ct \end{pmatrix}$$

It keeps the distance.

$$\begin{aligned} z'^2 - (ct')^2 &= (\gamma z - \beta\gamma ct)^2 - (-\beta\gamma z + \gamma ct)^2 \\ &= z^2(\gamma^2 - \beta^2\gamma^2) - (ct)^2(\gamma^2 - \beta^2\gamma^2) \\ &= z^2 - (ct)^2 \end{aligned}$$

unchanged. The inverse transformation is

$$\begin{pmatrix} z \\ ct \end{pmatrix} = \begin{pmatrix} \gamma & \beta\gamma \\ \beta\gamma & \gamma \end{pmatrix} \begin{pmatrix} z' \\ ct' \end{pmatrix}$$

$$\begin{pmatrix} \gamma & \beta\gamma \\ \beta\gamma & \gamma \end{pmatrix} \begin{pmatrix} \gamma & -\beta\gamma \\ -\beta\gamma & \gamma \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$$

(3) Time dilation:

$$\begin{pmatrix} \Delta z \\ c\Delta t \end{pmatrix} = \begin{pmatrix} \gamma & \beta\gamma \\ \beta\gamma & \gamma \end{pmatrix} \underbrace{\begin{pmatrix} \Delta z' \\ c\Delta t' \end{pmatrix}}_{\text{rest frame of particle}} \qquad \Delta z' = 0$$

measured at same position.

$$\Delta t = \gamma \, \Delta t'$$

(4) Length contraction:

Here we require that the length is measured simultaneously in the moving frame.

$$\begin{pmatrix} \Delta z' \\ c\Delta t' \end{pmatrix} = \begin{pmatrix} \gamma & -\beta\gamma \\ -\beta\gamma & \gamma \end{pmatrix} \underbrace{\begin{pmatrix} \Delta z \\ c\Delta t \end{pmatrix}}_{\text{moving frame}} \qquad c\Delta t = 0$$

(simultaneous measurement)

$$\Delta z' = \gamma \, \Delta z$$

(5) Velocity addition.

rest: 0 $1 \to \beta_1$ $2 \to \beta_2$

$$u_1 = L_{10}\, u_0 \qquad u_2 = L_{21}\, u_1 \quad \Rightarrow \quad L_{20} = L_{21} L_{10}$$

$$u_2 = L_{20}\, u_0$$

$$\begin{pmatrix} \gamma_{21} & \beta_{21}\gamma_{21} \\ \beta_{21}\gamma_{21} & \gamma_{21} \end{pmatrix} = \begin{pmatrix} \gamma_2 & \beta_2\gamma_2 \\ \beta_2\gamma_2 & \gamma_2 \end{pmatrix} \begin{pmatrix} \gamma_1 & -\beta_1\gamma_1 \\ -\beta_1\gamma_1 & \gamma_1 \end{pmatrix} = \begin{pmatrix} \gamma_1\gamma_2(1-\beta_1\beta_2) & \gamma_1\gamma_2(\beta_2-\beta_1) \\ \gamma_1\gamma_2(\beta_2-\beta_1) & \gamma_1\gamma_2(1-\beta_1\beta_2) \end{pmatrix}$$

$$\Rightarrow \quad \beta_{21} = \frac{\beta_2 - \beta_1}{1 - \beta_1\beta_2}$$

Electromagnetism in 4 vector notation

SET - 34B

1. $A^{\mu} = \left(\frac{1}{c}\phi, \vec{A}\right)$

2. $F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu$

$$
\begin{aligned}
F_{0i} &= \partial_0 A_i - \partial_i A_0 \\
&= \frac{1}{c}\frac{\partial}{\partial t} A_i - \nabla_i\left(-\frac{1}{c}\phi\right) \\
&= \frac{1}{c}\left[\frac{\partial A_i}{\partial t} + \vec{\nabla}\phi\right] \\
&= -\frac{1}{c} E_i
\end{aligned}
$$

$$
\begin{aligned}
F_{ij} &= \partial_i A_j - \partial_j A_i \\
&= \epsilon_{ijk} B_k
\end{aligned}
$$

$$
\begin{aligned}
\vec{B} &= \vec{\nabla} \times \vec{A} \\
B_i &= \epsilon_{ijk} \nabla_j A_k \\
\epsilon_{mni} B_i &= \epsilon_{imn}\epsilon_{ijk} \nabla_j A_k \\
&= \nabla_m A_n - \nabla_n A_m
\end{aligned}
$$

$$
F_{\mu\nu} = \begin{bmatrix}
0 & -\frac{1}{c}E_x & -\frac{1}{c}E_y & -\frac{1}{c}E_z \\
\frac{1}{c}E_x & 0 & B_z & -B_y \\
\frac{1}{c}E_y & -B_z & 0 & B_x \\
\frac{1}{c}E_z & B_y & -B_x & 0
\end{bmatrix}
$$

3. $F^{\mu\nu} = g^{\mu\alpha} F_{\alpha\beta} g^{\beta\nu}$

$$
\begin{aligned}
F^{0i} &= g^{0\alpha} F_{\alpha\beta} g^{\beta i} = g^{00} F_{0j} g^{ji} = -F_{0j} g^{ji} = \frac{1}{c} E^i \\
F^{ij} &= g^{i\alpha} F_{\alpha\beta} g^{\beta j} = g^{ii'} F_{i'j'} g^{j'j} = \epsilon^{ijk} B^k
\end{aligned}
$$

$$
F^{\mu\nu} = \begin{bmatrix}
0 & \frac{1}{c}E_x & \frac{1}{c}E_y & \frac{1}{c}E_z \\
-\frac{1}{c}E_x & 0 & B_z & -B_y \\
-\frac{1}{c}E_y & -B_z & 0 & B_x \\
-\frac{1}{c}E_z & B_y & -B_x & 0
\end{bmatrix}
$$

(4)

$$\partial_\mu F^{\mu 0} = \partial_i F^{i0} = \nabla_i \left(-\frac{1}{c}E_i\right) = -\frac{1}{c}\vec{\nabla}\cdot\vec{E} = -\frac{1}{c}\frac{1}{\epsilon_0}\rho$$

$$\partial_\mu F^{\mu i} = \partial_0 F^{0i} + \partial_j F^{ji}$$

$$= \frac{1}{c}\frac{\partial}{\partial t}\left(\frac{1}{c}E^i\right) + \nabla_j \epsilon^{jik} B^k$$

$$= \frac{1}{c^2}\frac{\partial E^i}{\partial t} - \epsilon^{ijk}\nabla_j B^k$$

$$= \mu_0 \epsilon_0 \frac{\partial E^i}{\partial t} - (\nabla\times\vec{B})^i$$

$$= -\mu_0 J^i$$

$$-\partial_\mu F^{\mu\nu} = \left(\frac{1}{c}\frac{1}{\epsilon_0}\rho ,\ \mu_0 \vec{J}\right)$$

$$= \mu_0\left(\frac{1}{c}\frac{1}{\mu_0\epsilon_0}\rho ,\ \vec{J}\right)$$

$$= \mu_0 (c\rho, \vec{J})$$

$$J^\nu = (c\rho, \vec{J})$$

$$= \mu_0 J^\nu$$

(5)

$$J^\mu A_\mu = J^0 A_0 + \vec{J}\cdot\vec{A}$$

$$= -J^0 A^0 + \vec{J}\cdot\vec{A}$$

$$= -\rho\phi + \vec{J}\cdot\vec{A}$$

$$= -(\rho\phi - \vec{J}\cdot\vec{A})$$

$$= -\text{Energy.}$$

6
$$\tilde{F}^{\mu\nu} = \frac{1}{2}\epsilon^{\mu\nu\alpha\beta}F_{\alpha\beta}.$$

$$\tilde{F}^{0i} = \frac{1}{2}\epsilon^{0i\alpha\beta}F_{\alpha\beta} = \frac{1}{2}\epsilon^{0ijk}F_{jk}.$$
$$= \frac{1}{2}\epsilon^{0ijk}\epsilon_{jkm}B_m.$$
$$= \frac{1}{2}\epsilon_{jki}\epsilon_{jkm}B_m.$$
$$= B_i$$

$$\tilde{F}^{ij} = \frac{1}{2}\epsilon^{ijk0}F_{k0} + \frac{1}{2}\epsilon^{ij0k}F_{0k}.$$
$$= \epsilon^{ijk0}F_{k0} = -\epsilon^{0ijk}F_{k0}$$
$$= -\epsilon^{ijk}\frac{1}{c}E_k = -\frac{1}{c}\epsilon_{ijk}E_k.$$

$$\tilde{F}^{\mu\nu} = \begin{bmatrix} 0 & B_x & B_y & B_z \\ -B_x & 0 & -\frac{1}{c}E_z & +\frac{1}{c}E_y \\ -B_y & +\frac{1}{c}E_z & 0 & -\frac{1}{c}E_x \\ -B_z & -\frac{1}{c}E_y & +\frac{1}{c}E_x & 0 \end{bmatrix}$$

7
$$\partial_\mu \tilde{F}^{\mu 0} = \partial_i \tilde{F}^{i0} = -\vec{\nabla}\cdot\vec{B} = 0.$$

$$\partial_\mu \tilde{F}^{\mu i} = \partial_0 \tilde{F}^{0i} + \partial_j \tilde{F}^{ji}$$
$$= \frac{1}{c}\frac{\partial}{\partial t}B_i + \nabla_j \frac{-1}{c}\epsilon^{jik}E^k.$$
$$= \frac{1}{c}\left[\frac{\partial B_i}{\partial t} + (\vec{\nabla}\times\vec{E})^i\right] = 0$$

8 $$u^\alpha = \frac{dx^\alpha}{ds} = \frac{d}{ds}(ct, \vec{x}) = \frac{dt}{ds}(c, \vec{v}) = r(c, \vec{v})$$

$$-ds^2 = -dt^2 + dx^2.$$
$$ds = dt\sqrt{1-\beta^2}$$
$$r = \frac{dt}{ds}.$$

9 $$F^{\alpha\beta} u_\beta$$

$$F^{0\beta} u_\beta = F^{0i} u_i = \frac{1}{c} E^i r v^i = \frac{r}{c} \vec{E}\cdot\vec{v}$$

$$F^{i\beta} u_\beta = F^{i0} u_0 + F^{ij} u_j = -\frac{1}{\cancel{c}} E^i (-r\cancel{c}) + \epsilon^{ijk} B^k v_j r = r\left[E^i + (\vec{v}\times\vec{B})^i\right]$$

$$p^\alpha = \left(\frac{E}{c}, \vec{p}\right)$$

10 $$f^\alpha = \frac{dp^\alpha}{ds} = \frac{dt}{ds}\left(\frac{1}{c}\frac{\partial E}{\partial t}, \frac{d\vec{p}}{dt}\right) = r\left(\frac{1}{c}\frac{\partial E}{\partial t}, \vec{F}\right)$$

11 $$q F^{\alpha\beta} u_\beta = \left(q \frac{r}{c} \vec{E}\cdot\vec{v}, r\vec{F}\right) = r\left(\frac{1}{c} \vec{F}\cdot\vec{v}, \vec{F}\right)$$

(12) Summary:

→ $A^\mu = \left(\frac{1}{c}\phi, \vec{A}\right)$

→ Maxwell stress tensor:

$$F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu$$

$$\begin{cases} \vec{E} = -\vec{\nabla}\phi - \dfrac{\partial \vec{A}}{\partial t} \\ \vec{B} = \vec{\nabla}\times\vec{A} \end{cases}$$

→ Inhomogeneous Maxwell's eqn:

$$\partial_\mu F^{\nu\mu} = \mu_0 J^\nu$$

$$\begin{cases} \vec{\nabla}\cdot\vec{E} = \dfrac{1}{\epsilon_0}\rho \\ \vec{\nabla}\times\vec{B} - \mu_0\epsilon_0 \dfrac{\partial \vec{E}}{\partial t} = \mu_0 \vec{J} \end{cases}$$

→ Homogeneous Maxwell's eqn:

$$\partial_\mu \tilde{F}^{\nu\mu} = 0$$

$$\tilde{F}^{\mu\nu} = \frac{1}{2}\epsilon^{\mu\nu\alpha\beta} F_{\alpha\beta}$$

$$\begin{cases} \vec{\nabla}\cdot\vec{B} = 0 \\ \vec{\nabla}\times\vec{E} + \dfrac{\partial \vec{B}}{\partial t} = 0 \end{cases}$$

→ Lorentz force:

$$f^\alpha = \frac{dp^\alpha}{ds} = q\, F^{\alpha\beta} u_\beta$$

$$\begin{cases} \vec{F} = q\left(\vec{E} + \vec{v}\times\vec{B}\right) \\ \dfrac{\partial E}{\partial t} = \vec{F}\cdot\vec{v} \end{cases}$$