SET-33

A point charge moving with constant speed.

① The fields in terms of the potentials,

$$\vec{B} = \vec{\nabla} \times \vec{A},$$

$$\vec{E} = -\vec{\nabla}\phi - \frac{\partial \vec{A}}{\partial t},$$

in the Lorenz gauge satisfy the non-homogeneous wave equations

$$\left(-\nabla^2 + \frac{1}{c^2}\frac{\partial^2}{\partial t^2}\right)\phi(\vec{r},t) = \frac{1}{\epsilon_0}\rho(\vec{r},t)$$

$$\left(-\nabla^2 + \frac{1}{c^2}\frac{\partial^2}{\partial t^2}\right)\vec{A}(\vec{r},t) = \mu_0 \vec{J}(\vec{r},t)$$

② Retarded Green's function:

$$\left(-\nabla^2 + \frac{1}{c^2}\frac{\partial^2}{\partial t^2}\right)\frac{1}{4\pi}G(\vec{r}-\vec{r}', t-t') = \delta^{(3)}(\vec{r}-\vec{r}')\,\delta(t-t')$$

with solution (satisfying causality)

$$G(\vec{r}-\vec{r}', t-t') = \frac{1}{|\vec{r}-\vec{r}'|}\,\delta\left(t-t'-\frac{1}{c}|\vec{r}-\vec{r}'|\right)$$

③ In terms of the Green's function we can solve

$$\phi(\vec{r},t) = \frac{1}{4\pi\epsilon_0}\int d^3r' \int_{-\infty}^{+\infty} dt'\; G(\vec{r}-\vec{r}', t-t')\,\rho(\vec{r}',t')$$

$$\vec{A}(\vec{r},t) = \frac{\mu_0}{4\pi}\int d^3r' \int_{-\infty}^{+\infty} dt'\; G(\vec{r}-\vec{r}', t-t')\,\vec{J}(\vec{r}',t')$$

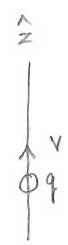
(4) <u>Lienard - Wiechert potentials :</u>

If we integrate over t' we obtain

$$\phi(\vec{r},t) = \frac{1}{4\pi\epsilon_0} \int d^3r' \frac{1}{|\vec{r}-\vec{r}'|} \rho\left(\vec{r}', t - \frac{1}{c}|\vec{r}-\vec{r}'|\right)$$

$$\vec{A}(\vec{r},t) = \frac{\mu_0}{4\pi} \int d^3r' \frac{1}{|\vec{r}-\vec{r}'|} \vec{J}\left(\vec{r}', t - \frac{1}{c}|\vec{r}-\vec{r}'|\right).$$

(5) A point charge moving with constant speed is described by



$$\vec{r}_a(t) = x_a(t)\,\hat{i} + y_a(t)\,\hat{j} + z_a(t)\,\hat{k}$$
$$= 0\,\hat{i} + 0\,\hat{j} + vt\,\hat{k}$$

$$\vec{V}_a(t) = \frac{d}{dt}\vec{r}_a(t)$$
$$= 0\,\hat{i} + 0\,\hat{j} + v\,\hat{k}$$

$$\rho(\vec{r}',t') = q\,\delta(x'-x_a(t'))\,\delta(y'-y_a(t'))\,\delta(z'-z_a(t'))$$
$$= q\,\delta(x')\,\delta(y')\,\delta(z'-vt')$$

$$\vec{J}(\vec{r}',t') = \vec{V}_a(t')\,\rho(\vec{r}',t')$$
$$= \hat{z}\,qv\,\delta(x')\,\delta(y')\,\delta(z'-vt')$$

6 Using 5 in 3 we have the scalar potential,

$$\phi(\vec{r},t) = \frac{1}{4\pi\epsilon_0} \int d^3r' \int_{-\infty}^{+\infty} dt' \; G(\vec{r}-\vec{r}', t-t') \; \rho(\vec{r}',t')$$

$$= \frac{q}{4\pi\epsilon_0} \int d^3r' \int_{-\infty}^{+\infty} dt' \; \frac{\delta\left(t-t'-\frac{1}{c}|\vec{r}-\vec{r}'|\right)}{|\vec{r}-\vec{r}'|} \; \delta(x')\,\delta(y')\,\delta(z'-vt')$$

$$= \frac{q}{4\pi\epsilon_0} \int_{-\infty}^{+\infty} dt' \int_{-\infty}^{+\infty} dx' \int_{-\infty}^{+\infty} dy' \int_{-\infty}^{+\infty} dz' \; \frac{\delta\left(t-t'-\frac{1}{c}\sqrt{(x-x')^2+(y-y')^2+(z-z')}\right)}{\sqrt{(x-x')^2+(y-y')^2+(z-z')^2}} \; \delta(x')\,\delta(y')\,\delta(z'-vt')$$

$$= \frac{q}{4\pi\epsilon_0} \int_{-\infty}^{+\infty} dt' \; \frac{\delta\left(t-t'-\frac{1}{c}\sqrt{x^2+y^2+(z-vt')^2}\right)}{\sqrt{x^2+y^2+(z-vt')^2}}$$

7 This involves an integral of the kind

$$\int_{-\infty}^{+\infty} dx \; \delta(F(x)) \, .$$

We shall digress to evaluate this integral.

8 Let us begin by evaluating

$$\int_{-\infty}^{+\infty} dx\ \delta(kx) = \frac{1}{k} \int_{-\infty}^{+\infty} dx'\ \delta(x') \qquad (x' = kx\ ,\ k > 0)$$

$$= \frac{1}{k}$$

$$\int_{-\infty}^{+\infty} dx\ \delta(kx) = \frac{1}{k} \int_{\infty}^{-\infty} dx'\ \delta(x') \qquad (x' = kx\ ,\ k < 0)$$

$$= -\frac{1}{k}$$

Together, we can write

$$\int_{-\infty}^{+\infty} dx\ \delta(kx) = \frac{1}{|k|}\ ,$$

which is stated, before completing the x' integral as.

$$\int_{-\infty}^{+\infty} dx\ \delta(kx) = \frac{1}{|k|} \int_{-\infty}^{+\infty} dx\ \delta(x)\ ,$$

or

$$\delta(kx) = \frac{\delta(x)}{|k|}\ .$$

9 Let us now evaluate

$$\int_{-\infty}^{+\infty} dx\ \delta\big((x-a_1)(x-a_2)\big) = \int_{a_1-\delta}^{a_1+\delta} dx\ \delta\big((x-a_1)(x-a_2)\big) + \int_{a_2-\delta}^{a_2+\delta} dx\ \delta\big((x-a_1)(x-a_2)\big)$$

$$= \int_{a_1-\delta}^{a_1+\delta} dx.\ \delta\big((x-a_1)(a_1-a_2)\big) + \int_{a_2-\delta}^{a_2+\delta} dx\ \delta\big((a_2-a_1)(x-a_2)\big)$$

$$= \frac{1}{|a_1-a_2|} + \frac{1}{|a_2-a_1|} \qquad \text{(using 8)}$$

Thus, we conclude

$$\delta\big((x-a_1)(x-a_2)\big) = \frac{\delta(x-a_1)}{|a_1-a_2|} + \frac{\delta(x-a_2)}{|a_2-a_1|}$$

10 We can immediately generalize the discussion in 9 & write

$$\int_{-\infty}^{+\infty} dx\ \delta\big((x-a_1)\ldots(x-a_n)\big) = \sum_{i=1}^{n} \frac{\delta(x-a_i)}{\left| \left.\dfrac{(x-a_1)\ \ldots\ (x-a_n)}{(x-a_i)}\right|_{x=a_i} \right|}$$

11 Next, we make the observation that if

$$f(x) = a_0\ (x-a_1)(x-a_2)(x-a_3)$$

then

$$\frac{df}{dx} = a_0\ (x-a_2)(x-a_3) + a_0(x-a_1)(x-a_3) + a_0(x-a_1)(x-a_2)$$

$$= \sum_{i=1}^{3} \frac{f(x)}{(x-a_i)}$$

(12) In general we have.

$$\frac{df}{dx} = \sum_r \frac{f(x)}{(x-a_r)} \qquad r - \text{roots of } f(x)=0 .$$

(13) Using (12) in (10) we thus conclude that

$$\int_{-\infty}^{+\infty} dx \; \delta(F(x)) = \sum_r \frac{\delta(x-a_r)}{\left| \frac{\partial F}{\partial x}\Big|_{x=a_r} \right|}$$

(14) Using (13) in (6) we have.

$$\phi(\vec{r},t) = \frac{q}{4\pi\epsilon_0} \int_{-\infty}^{+\infty} dt' \frac{1}{\sqrt{x^2+y^2+(z-vt')^2}} \sum_r \frac{\delta(t'-t_r)}{\left| \frac{\partial F}{\partial t'}\Big|_{t'=t_r} \right|}$$

where

$$F(t') = t - t' - \frac{1}{c}\sqrt{x^2+y^2+(z-vt')^2}$$

and t_r are the roots of the equation

$$F(t') = 0 ,$$

which is a quadratic equation — thus has two solutions.

15 Using 14 we have.

$$\frac{\partial F}{\partial t'} = -1 + \frac{v}{c}\frac{(z-vt')}{\sqrt{x^2+y^2+(z-vt')^2}}$$

$$\left|\frac{\partial F}{\partial t'}\right| = 1 - \frac{v}{c}\frac{(z-vt')}{\sqrt{x^2+y^2+(z-vt')^2}} \qquad \left(\because \frac{v}{c}<1 \text{ and } \frac{x}{\sqrt{a^2+x^2}}<1\right)$$

16 Using 15 in 14

$$\phi(\vec{r},t) = \frac{q}{4\pi\epsilon_0}\sum_{r=1}^{2}\frac{1}{\sqrt{x^2+y^2+(z-vt_r)^2} - \frac{v}{c}(z-vt_r)}$$

$$= \frac{q}{4\pi\epsilon_0}\sum_{r=1}^{2}\frac{1}{c(t-t_r) - \frac{v}{c}(z-vt_r)} \qquad (\text{using } F(t_r)=0)$$

$$= \frac{q}{4\pi\epsilon_0}\sum_{r=1}^{2}\frac{1}{c}\frac{1}{\left(t-\frac{v}{c}\frac{z}{c}\right) - t_r\left(1-\frac{v^2}{c^2}\right)}$$

17 Thus the problem reduces to finding the roots of

$$F(t') = t - t' - \frac{1}{c}\sqrt{x^2+y^2+(z-vt')^2} = 0$$

$$t_r^2\left(1-\frac{v^2}{c^2}\right) - 2t_r\left(t-\frac{v}{c}\frac{z}{c}\right) + t^2 - \frac{x^2+y^2}{c^2} - \frac{z^2}{c^2} = 0$$

$$t_r = \frac{\left(t-\frac{v}{c}\frac{z}{c}\right) \pm \sqrt{\left(t-\frac{v}{c}\frac{z}{c}\right)^2 - \cancel{4}\left(1-\frac{v^2}{c^2}\right)\left(t^2-\frac{x^2+y^2}{c^2}-\frac{z^2}{c^2}\right)}}{\left(1-\frac{v^2}{c^2}\right)}$$

(18) $$\left(\sqrt{}\right)^2 = t^2 + \frac{v^2}{c^2}\frac{z^2}{c^2} - 2\frac{v}{c}\frac{z}{c}t - \left(1-\frac{v^2}{c^2}\right)\left(t^2 - \frac{x^2+y^2}{c^2} - \frac{z^2}{c^2}\right)$$

$$= -2\frac{v}{c}\frac{z}{c}t + \frac{z^2}{c^2} + \frac{v^2}{c^2}t^2 + \left(1-\frac{v^2}{c^2}\right)\frac{x^2+y^2}{c^2}$$

$$= \frac{1}{c^2}(z-vt)^2 + \frac{1}{c^2}\left(1-\frac{v^2}{c^2}\right)(x^2+y^2)$$

$z - vt \rightarrow$ retardation
$1 - \frac{v^2}{c^2} \rightarrow$ Lorentz contraction.

(19) Using (18) in (17)

$$t_r = \frac{\left(t - \frac{v}{c}\frac{z}{c}\right) \pm \frac{1}{c}\sqrt{(x^2+y^2)\left(1-\frac{v^2}{c^2}\right) + (z-vt)^2}}{\left(1-\frac{v^2}{c^2}\right)}$$

(20) To investigate which root contributes set $v=0$.

$$t_r = t \pm \frac{1}{c}\sqrt{x^2+y^2+z^2}$$

Thus, retardation (or causal solution) corresponds to negative.

(21) Using (19) and (20) we conclude.

$$t_r = \frac{\left(t - \frac{v}{c}\frac{z}{c}\right) - \frac{1}{c}\sqrt{(x^2+y^2)\left(1-\frac{v^2}{c^2}\right) + (z-vt)^2}}{\left(1-\frac{v^2}{c^2}\right)}$$

which can be rewritten as.

$$\left(t - \frac{v}{c}\frac{z}{c}\right) - t_r\left(1-\frac{v^2}{c^2}\right) = \frac{1}{c}\sqrt{(x^2+y^2)\left(1-\frac{v^2}{c^2}\right) + (z-vt)^2}$$

22. Using 21 in 16 we have, collecting only negative root,

$$\phi(\vec{r},t) = \frac{1}{4\pi\epsilon_0} \frac{q}{\sqrt{(x^2+y^2)(1-\frac{v^2}{c^2}) + (z-vt)^2}}$$

→ $(z-vt)$ corresponds to retardation

→ $1-\frac{v^2}{c^2}$ factors are relativistic contributions — length contraction.

23. Since the vector potential satisfies almost identical form we have.

$$\vec{A}(\vec{r},t) = \frac{\mu_0}{4\pi} \frac{q v \hat{z}}{\sqrt{(x^2+y^2)(1-\frac{v^2}{c^2}) + (z-vt)^2}}$$

24. We can now evaluate the fields.

$$\vec{E}(\vec{r},t) = -\vec{\nabla}\phi(\vec{r},t) - \frac{\partial}{\partial t}\vec{A}(\vec{r},t)$$

25. Using 23 $\qquad \beta = \frac{v}{c}$

$$\frac{\partial}{\partial t}\vec{A}(\vec{r},t) = \frac{\mu_0}{4\pi} \hat{z}\, v^2 \frac{q\,(z-vt)}{\left[(x^2+y^2)(1-\beta^2) + (z-vt)^2\right]^{\frac{3}{2}}}$$

$$= \frac{1}{4\pi\epsilon_0} \hat{z}\, \beta^2 \frac{q\,(z-vt)}{\left[(x^2+y^2)(1-\beta^2) + (z-vt)^2\right]^{\frac{3}{2}}}$$

26 Using 22

$$\vec{\nabla}\phi(\vec{r},t) = -\frac{q}{4\pi\epsilon_0}\frac{x(1-\beta^2)\,\hat{i} + y(1-\beta^2)\,\hat{j} + (z-vt)\,\hat{k}}{\left[(x^2+y^2)(1-\beta^2) + (z-vt)^2\right]^{\frac{3}{2}}}$$

27 Using 25 and 26 in 24

$$\vec{E}(\vec{r},t) = \frac{q}{4\pi\epsilon_0}(1-\beta^2)\frac{x\,\hat{i} + y\,\hat{j} + (z-vt)\,\hat{k}}{\left[(x^2+y^2)(1-\beta^2) + (z-vt)^2\right]^{\frac{3}{2}}}$$

28

$$\vec{B}(\vec{r},t) = \vec{\nabla}\times\vec{A}(\vec{r},t)$$

$$= \vec{\nabla}\times\frac{\mu_0}{4\pi}\frac{qv\,\hat{z}}{\sqrt{(x^2+y^2)(1-\beta^2)+(z-vt)^2}}$$

$$= -\frac{\mu_0}{4\pi}qv\,\hat{z}\times\vec{\nabla}\frac{1}{\sqrt{(x^2+y^2)(1-\beta^2)+(z-vt)^2}}$$

$$= \frac{\mu_0}{4\pi}qv\,\hat{k}\times\frac{x(1-\beta^2)\,\hat{i} + y(1-\beta^2)\,\hat{j} + (z-vt)\,\hat{k}}{\left[(x^2+y^2)(1-\beta^2) + (z-vt)^2\right]^{\frac{3}{2}}}$$

$$= \frac{\mu_0}{4\pi}qv(1-\beta^2)\frac{\sqrt{x^2+y^2}\;\hat{\phi}}{\left[(x^2+y^2)(1-\beta^2) + (z-vt)^2\right]^{\frac{3}{2}}}$$

$$\hat{\phi} = \frac{x\,\hat{j} - y\,\hat{i}}{\sqrt{x^2+y^2}}$$

(29) Summary:

$$\phi(\vec{r},t) = \frac{q}{4\pi\epsilon_0} \frac{1}{\sqrt{(x^2+y^2)(1-\beta^2) + (z-vt)^2}}$$

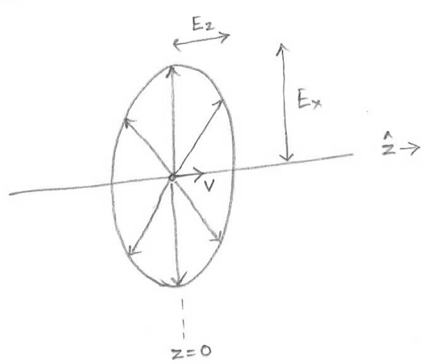
$$\vec{A}(\vec{r},t) = \frac{1}{c} \frac{q}{4\pi\epsilon_0} \frac{v}{c} \frac{1}{\sqrt{(x^2+y^2)(1-\beta^2) + (z-vt)^2}}$$

$$\vec{E}(\vec{r},t) = (1-\beta^2) \frac{q}{4\pi\epsilon_0} \frac{x\hat{i} + y\hat{j} + (z-vt)\hat{k}}{\left[(x^2+y^2)(1-\beta^2) + (z-vt)^2\right]^{\frac{3}{2}}}$$

$$\hat{\phi} = \frac{x\hat{j} - y\hat{i}}{\sqrt{x^2+y^2}}$$

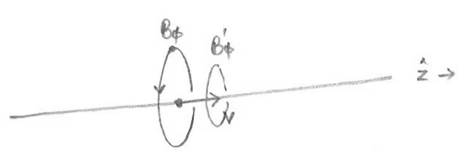
$$\vec{B}(\vec{r},t) = \frac{1}{c} \beta (1-\beta^2) \frac{q}{4\pi\epsilon_0} \frac{(x\hat{j} - y\hat{i})}{\left[(x^2+y^2)(1-\beta^2) + (z-vt)^2\right]^{\frac{3}{2}}}$$

(30) Snapshot at $t=0$.



$$E_x = |\vec{E}(x,0,0,0)| = \frac{1}{\sqrt{1-\beta^2}} \frac{q}{4\pi\epsilon_0} \frac{1}{x^2}$$

$$E_z = |\vec{E}(0,0,z,0)| = (1-\beta^2) \frac{q}{4\pi\epsilon_0} \frac{1}{z^2}$$



$$cB_\phi = |c\vec{B}(x,0,0,0)| = \frac{\beta}{\sqrt{1-\beta^2}} \frac{q}{4\pi\epsilon_0} \frac{1}{x^2}$$

$$cB'_\phi = |c\vec{B}(x,0,z,0)|$$

$$= \beta(1-\beta^2) \frac{q}{4\pi\epsilon_0} \frac{x}{\left[x^2(1-\beta^2) + z^2\right]^{\frac{3}{2}}}$$