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Retarded Green's function

① Consider the Green's function equation

$$\left(-\nabla^2 + \frac{1}{c^2}\frac{\partial^2}{\partial t^2}\right) G(\vec{r}-\vec{r}\,', t-t') = \delta^{(3)}(\vec{r}-\vec{r}\,')\,\delta(t-t')$$

② Fourier transforming in the time

$$G(\vec{r}-\vec{r}\,', t-t') = \int_{-\infty}^{+\infty} \frac{d\omega}{2\pi}\, e^{-i\omega(t-t')}\, \tilde{G}(\vec{r}-\vec{r}\,', \omega)$$

$$\tilde{G}(\vec{r}-\vec{r}\,', \omega) = \int_{-\infty}^{+\infty} dt\, e^{i\omega(t-t')}\, G(\vec{r}-\vec{r}\,', t-t')$$

③ Using ② in ① we have.

$$\left(-\nabla^2 - \frac{\omega^2}{c^2}\right) \tilde{G}(\vec{r}-\vec{r}\,', \omega) = \delta^{(3)}(\vec{r}-\vec{r}\,')$$

④ We can choose $\vec{r}\,'=0$ without loss of generality.

$$-\left(\nabla^2 + \frac{\omega^2}{c^2}\right) \tilde{G}(\vec{r}, \omega) = \delta^{(3)}(\vec{r})$$

which renders $\tilde{G}$ to be spherically symmetric.

$$-\left(\frac{1}{r^2}\frac{d}{dr} r^2 \frac{d}{dr} + \frac{\omega^2}{c^2}\right) \tilde{G}(r, \omega) = \delta^{(3)}(\vec{r})$$

5 Integrating w.r.t. θ and ϕ on both sides we have.

$$-\left(\frac{1}{r^2}\frac{d}{dr}r^2\frac{d}{dr}+\frac{\omega^2}{c^2}\right)\tilde{G}(r,\omega)=\frac{\delta(r)}{4\pi r^2}$$

6 Integrating 3, or 5, on a sphere of negligible radius we obtain the condition

$$\underset{r\to 0}{Lt}\ r^2\frac{d}{dr}\tilde{G}(r,\omega)+\underset{r\to 0}{Lt}\ \frac{\omega^2}{c^2}\int_r d^3r'\ \tilde{G}(r',\omega)=-\frac{1}{4\pi}$$

7 For $\omega=0$ (static case) we have.

$$\tilde{G}(r,0)=\frac{1}{4\pi r}$$

Thus, we can argue that

$$\underset{r\to 0}{Lt}\ \tilde{G}(r,\omega)\sim\frac{1}{r}.$$

Thus,

$$\underset{r\to 0}{Lt}\int_r d^3r'\ \tilde{G}(r,\omega)=\underset{r\to 0}{Lt}\ 4\pi\int_0^r dr'\ r'^2\frac{1}{r'}=0.$$

8 Thus, we need to solve:

$$-\left(\frac{1}{r^2}\frac{d}{dr}r^2\frac{d}{dr}+\frac{\omega^2}{c^2}\right)\tilde{G}(r,\omega)=\frac{\delta(r)}{4\pi r^2}$$

for continuity conditions:

(i) $$\underset{r\to 0}{Lt}\ r^2\frac{d}{dr}\tilde{G}(r,\omega)=-\frac{1}{4\pi}$$

9 Let $\tilde{G}(r,\omega) = \frac{1}{r} g(r,\omega)$

$$\frac{1}{r^2}\frac{d}{dr} r^2 \frac{d}{dr}\tilde{G} = \frac{2}{r}\frac{d}{dr}\tilde{G} + \frac{d^2}{dr^2}\tilde{G}$$

$$= \frac{2}{r}\frac{d}{dr}\frac{g}{r} + \frac{d^2}{dr^2}\frac{g}{r}$$

$$= -\frac{2}{r^3} g + \frac{2}{r^2}\frac{dg}{dr} + \frac{d}{dr}\left(-\frac{1}{r^2} g + \frac{1}{r}\frac{dg}{dr}\right)$$

$$= -\frac{2}{r^3} g + \frac{2}{r^2}\frac{dg}{dr} + \frac{2}{r^3} g - \frac{1}{r^2}\frac{dg}{dr} - \frac{1}{r^2}\frac{dg}{dr} + \frac{1}{r}\frac{d^2 g}{dr^2}.$$

✓ ✓ ✓ ✓ ✓

10 Using 9 in 8

$$\left(-\frac{d^2}{dr^2} - \frac{\omega^2}{c^2}\right) g(r,\omega) = \frac{\delta(r)}{4\pi r} \qquad \text{— (i)}$$

and

$$\underset{r\to 0}{Lt}\; r^2 \frac{d}{dr}\tilde{G}(r,\omega) = -\frac{1}{4\pi} \qquad \text{— (ii)}$$

11 For $r \neq 0$ we have.

$$\frac{d^2}{dr^2} g(r,\omega) = -\frac{\omega^2}{c^2} g(r,\omega)$$

$$\Rightarrow \quad g(r,\omega) = \frac{A}{4\pi} e^{i\frac{\omega}{c}r} + \frac{B}{4\pi} e^{-i\frac{\omega}{c}r} \qquad \text{for } r > 0.$$

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$$\frac{d}{dr}\tilde{G}(r,\omega) = \frac{d}{dr}\frac{1}{r} g$$

$$= -\frac{1}{r^2} g + \frac{1}{r}\frac{d}{dr} g$$

$$= -\frac{A}{4\pi r^2} e^{i\frac{\omega}{c}r} - \frac{B}{4\pi r^2} e^{-i\frac{\omega}{c}r} + \frac{1}{r}\left(\frac{A}{4\pi} i\frac{\omega}{c} e^{i\frac{\omega}{c}r} - \frac{B}{4\pi} i\frac{\omega}{c} e^{-i\frac{\omega}{c}r}\right)$$

(13) Using (12) in (10)

$$\lim_{r\to 0} r^2\left[-\frac{A}{4\pi r^2}e^{i\frac{\omega}{c}r}-\frac{B}{4\pi r^2}e^{-i\frac{\omega}{c}r}+\frac{1}{r}\frac{iA}{4\pi}\frac{\omega}{c}e^{i\frac{\omega}{c}r}-\frac{1}{r}\frac{iB}{4\pi}\frac{\omega}{c}e^{-i\frac{\omega}{c}r}\right]=-\frac{1}{4\pi}$$

$$\Rightarrow \quad A+B=1.$$

(14) Thus, we have the Fourier transformed solution

$$\tilde{G}(r,\omega)=\frac{A}{4\pi r}e^{i\frac{\omega}{c}r}+\frac{B}{4\pi r}e^{-i\frac{\omega}{c}r}, \qquad A+B=1.$$

or

$$\tilde{G}(\vec{r}-\vec{r}',\omega)=\frac{A}{4\pi|\vec{r}-\vec{r}'|}e^{i\frac{\omega}{c}|\vec{r}-\vec{r}'|}+\frac{B}{4\pi|\vec{r}-\vec{r}'|}e^{-i\frac{\omega}{c}|\vec{r}-\vec{r}'|}, \qquad A+B=1.$$

(15) Using (14) in (2)

$$G(\vec{r}-\vec{r}',t-t')=\int_{-\infty}^{+\infty}\frac{d\omega}{2\pi}e^{-i\omega(t-t')}\tilde{G}(\vec{r}-\vec{r}',\omega)$$

$$=\int_{-\infty}^{+\infty}\frac{d\omega}{2\pi}e^{-i\omega(t-t')}\left[\frac{A}{4\pi|\vec{r}-\vec{r}'|}e^{i\frac{\omega}{c}|\vec{r}-\vec{r}'|}+\frac{B}{4\pi|\vec{r}-\vec{r}'|}e^{-i\frac{\omega}{c}|\vec{r}-\vec{r}'|}\right]$$

$$=\frac{1}{4\pi|\vec{r}-\vec{r}'|}\left[A\int_{-\infty}^{+\infty}\frac{d\omega}{2\pi}e^{-i\omega(t-t')+i\frac{\omega}{c}|\vec{r}-\vec{r}'|}+B\int_{-\infty}^{+\infty}\frac{d\omega}{2\pi}e^{-i\omega(t-t')-i\frac{\omega}{c}|\vec{r}-\vec{r}'|}\right]$$

$$=\frac{1}{4\pi|\vec{r}-\vec{r}'|}\left[A\,\delta\left((t-t')-\frac{1}{c}|\vec{r}-\vec{r}'|\right)+B\,\delta\left((t-t')+\frac{1}{c}|\vec{r}-\vec{r}'|\right)\right]$$

$A+B=1.$

16 A-term $t = t' + \frac{1}{c}|\vec{r}-\vec{r}'|$ $\rightarrow$ causal

B-term $t = t' - \frac{1}{c}|\vec{r}-\vec{r}'|$ $\rightarrow$ not causal

We require our process to be causal. This requires $B=0$ and $A+B=1$ $\Rightarrow$ $A=1$. Thus,

$$G(\vec{r}-\vec{r}', t-t') = \frac{1}{4\pi}\frac{1}{|\vec{r}-\vec{r}'|}\,\delta\left(t-t'-\frac{1}{c}|\vec{r}-\vec{r}'|\right)$$