SET-29

Magnetostatic energy

(1) $\vec{F} = -\vec{\nabla} \underbrace{W}_{\text{energy}}$

$\vec{E} = -\vec{\nabla}\phi - \frac{\partial \vec{A}}{\partial t}$

$\vec{B} = \vec{\nabla}\times\vec{A}$

(2)
$$\begin{aligned}
\vec{F} &= q\vec{E} + q\,\vec{v}\times\vec{B} \\
&= q\left(-\vec{\nabla}\phi - \frac{\partial \vec{A}}{\partial t}\right) + q\,\vec{v}\times(\vec{\nabla}\times\vec{A}) \\
&= -q\,\vec{\nabla}\phi - q\frac{\partial \vec{A}}{\partial t} + q\,\vec{\nabla}(\vec{v}\cdot\vec{A}) - q\,\vec{v}\cdot\vec{\nabla}\,\vec{A}
\end{aligned}$$

Note that $\vec{v}_a = \frac{d\vec{x}_a}{dt}$ $\quad \vec{\nabla}\cdot\vec{v}_a = \frac{\partial}{\partial \vec{x}}\cdot\frac{d\vec{x}_a}{dt} = 0$.

$$\vec{F} = -\vec{\nabla}(q\phi - q\,\vec{v}\cdot\vec{A}) - q\underbrace{\left(\frac{\partial}{\partial t} + \vec{v}\cdot\vec{\nabla}\right)}_{\frac{d}{dt}}\vec{A}$$

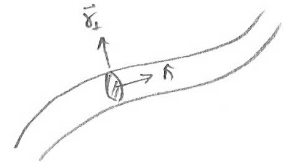
$$\frac{d\vec{p}}{dt} = -\vec{\nabla}(q\phi - q\,\vec{v}\cdot\vec{A}) - q\frac{d\vec{A}}{dt}$$

$$\frac{d}{dt}\underbrace{(\vec{p} + q\vec{A})}_{\text{conjugate momentum}} = -\vec{\nabla}(\underbrace{q\phi}_{\text{electrostatic energy}} - \underbrace{q\,\vec{v}\cdot\vec{A}}_{\text{magnetostatic energy}})$$

(3)
$$\begin{aligned}
W_m &= -q\,\vec{v}\cdot\vec{A} \\
&= -\int d^3r\, \vec{J}_1(\vec{r})\cdot\vec{A}(\vec{r}) \\
&= -\frac{\mu_0}{4\pi}\int d^3r \int d^3r'\, \frac{\vec{J}_1(\vec{r})\cdot\vec{J}_2(\vec{r}')}{|\vec{r}-\vec{r}'|}
\end{aligned}$$

4 $$\int d^3r \; \vec{J}(\vec{r}) = \int d^2r_\perp \int d\xi \; \vec{J}(\vec{r}_\perp, \xi)$$

$$= I \int d\xi \; \hat{n}(\xi)$$



5 Thus,

$$W_m = - I_1 \int d\xi \; \hat{n}_1(\xi) \cdot \vec{A}(\xi)$$

$$= - \frac{\mu_0}{4\pi} I_1 I_2 \int d\xi_1 \int d\xi_2 \; \frac{\hat{n}_1(\xi_1) \cdot \hat{n}_2(\xi_2)}{|\vec{r} - \vec{r}'|}$$

6 Magnetostatic energy of a closed loop of wire:

$$W_m = - I \oint d\xi \; \hat{n}(\xi) \cdot \vec{A}$$

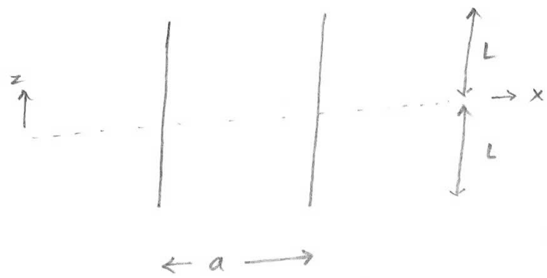
$$= - I \oint d\vec{l} \cdot \vec{A}$$

$$= - I \int_S d\vec{s} \cdot \vec{\nabla} \times \vec{A}$$

$$= - I \int_S d\vec{s} \cdot \vec{B}$$

$$= - I \; \Phi \; \text{flux}$$

7 Example : To parallel wires of finite length $2L$



8 $$W_m = -\frac{\mu_0}{4\pi} I_1 I_2 \int_{-L}^{L} dz \int_{-L}^{L} dz' \frac{\overbrace{\hat{z}\cdot\hat{z}}^{=1}}{\sqrt{\underbrace{(x-x')^2+(y-y')^2}_{=a^2}+(z-z')^2}}$$

$$= -\frac{\mu_0}{4\pi} I_1 I_2 \int_{-L}^{L} dz \int_{-L}^{L} dz' \frac{1}{\sqrt{a^2+(z-z')^2}}$$

$$z'-z = z'' \qquad dz' = dz''$$

$$= -\frac{\mu_0}{4\pi} I_1 I_2 \int_{-L}^{L} dz \int_{-L-z}^{L-z} dz'' \frac{1}{\sqrt{a^2+z''^2}}$$

$$z'' = a \sinh\theta \qquad dz'' = a \cosh\theta \cdot d\theta$$

$$= -\frac{\mu_0}{4\pi} I_1 I_2 \int_{-L}^{L} dz \int_{\sinh^{-1}\left(\frac{-L-z}{a}\right)}^{\sinh^{-1}\left(\frac{L-z}{a}\right)} \frac{a\cosh\theta\, d\theta}{a\cosh\theta}$$

$$= -\frac{\mu_0}{4\pi} I_1 I_2 \int_{-L}^{L} dz \left[\sinh^{-1}\left(\frac{L-z}{a}\right) + \sinh^{-1}\left(\frac{L+z}{a}\right) \right]$$

(9) Substitute $\frac{L-z}{a} = t$ and $\frac{L+z}{a} = t$ in the two terms

$-dz = a\,dt$ $\qquad$ $dz = a\,dt$

$$W_m = -\frac{\mu_0}{4\pi} I_1 I_2 \left[\int_{\frac{2L}{a}}^{0} (-a\,dt)\, \sinh^{-1} t + \int_{0}^{\frac{2L}{a}} a\,dt\, \sinh^{-1} t \right.$$

$$= -\frac{\mu_0}{4\pi} I_1 I_2 \, a \, 2 \int_0^{\frac{2L}{a}} dt\, \sinh^{-1} t$$

(10) Thus, magnetostatic energy per unit length is

$$\frac{W_m}{2L} = -\frac{\mu_0}{4\pi} 2 I_1 I_2 \frac{a}{2L} \int_0^{\frac{2L}{a}} dt\, \sinh^{-1} t$$

$y = \sinh^{-1} t$
$\sinh y = t$
$\cosh y\, dy = dt$

(11) $$\int_0^{\frac{2L}{a}} dt\, \sinh^{-1} t = \int_0^{\sinh^{-1}\frac{2L}{a}} \cosh y\, dy\, y$$

$$= \int_0^{\sinh^{-1}\frac{2L}{a}} dy\, y \cosh y$$

$$= \int_0^{\sinh^{-1}\frac{2L}{a}} dy\, y \frac{d}{dy} \sinh y$$

$$= \int_0^{\sinh^{-1}\frac{2L}{a}} dy \frac{d}{dy}\left(y \sinh y \right) - \int_0^{\sinh^{-1}\frac{2L}{a}} dy \sinh y$$

$$= \left(y \sinh y \right) \Big|_{y=0}^{y=\sinh^{-1}\frac{2L}{a}} - \sqrt{1 + \sinh^2 y}\, \Big|_{y=0}^{y=\sinh^{-1}\frac{2L}{a}}$$

$$= \frac{2L}{a} \sinh^{-1} \frac{2L}{a} - \sqrt{1 + \left(\frac{2L}{a}\right)^2} + 1$$

(12) Using $\quad \sinh^{-1} x = \ln\left(x + \sqrt{1+x^2}\right)$

$$\frac{W_m}{2L} = -\frac{\mu_0}{4\pi}\, 2 I_1 I_2\, \frac{a}{2L}\left[\frac{2L}{a} \ln\left\{\frac{2L}{a} + \sqrt{1+\left(\frac{2L}{a}\right)^2}\right\} - \sqrt{1+\left(\frac{2L}{a}\right)^2} + 1\right]$$

$$= -\frac{\mu_0}{4\pi}\, 2 I_1 I_2 \left[\ln\frac{2L}{a} + \ln\left\{1 + \sqrt{1+\left(\frac{a}{2L}\right)^2}\right\} - \sqrt{1+\left(\frac{a}{2L}\right)^2} + \frac{a}{2L}\right]$$

(13) For $a << 2L$ we have.

$$\frac{W_m}{2L} \xrightarrow{a<<2L} -\frac{\mu_0}{4\pi}\, 2 I_1 I_2 \left[\ln\frac{2L}{a} + \ln 2 - 1\right]$$

(15) $$\frac{F}{2L} = -\frac{\partial}{\partial a}\frac{W_m}{2L}$$

$$= -\frac{\mu_0}{4\pi}\frac{2 I_1 I_2}{a}$$

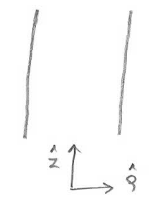
(16) We should be able to derive the force independently using.

$$\vec{F} = \int d^3r\ \vec{J}_1(\vec{r}) \times \vec{B}(\vec{r})$$

$$= \frac{\mu_0}{4\pi}\int d^3r \int d^3r'\ \vec{J}_1(\vec{r}) \times \left[\vec{J}_2(\vec{r}') \times \frac{(\vec{r}-\vec{r}')}{|\vec{r}-\vec{r}'|^3}\right]$$

(17) For parallel wires, both pointing in $\hat{z}$, we have.

$$\vec{F} = \frac{\mu_0}{4\pi} I_1 I_2 \int_{-L}^{L} dz \int_{-L}^{L} dz' \frac{\hat{z} \times [\hat{z} \times (\vec{r} - \vec{r}')]}{[a^2 + (z-z')^2]^{\frac{3}{2}}}$$



(18) $$\hat{z} \times [\hat{z} \times (\vec{r} - \vec{r}')] = \hat{z} \times [\hat{z} \times \{ a\hat{\rho} + (z-z')\hat{z} \}]$$

$$= \hat{z} \times (\hat{z} \times \hat{\rho})\, a$$

$$= -a\hat{\rho}$$

(19) $$\vec{F} = -\hat{\rho} \frac{\mu_0}{4\pi} I_1 I_2\, a \int_{-L}^{L} dz \int_{-L}^{L} dz' \frac{1}{[a^2 + (z-z')^2]^{\frac{3}{2}}}$$

(20) Contending to check for $L \to \infty$, we have.

$$\vec{F} = -\hat{\rho} \frac{\mu_0}{4\pi} I_1 I_2\, a \int_{-\infty}^{+\infty} dz \int_{-\infty}^{+\infty} dz' \frac{1}{[a^2 + (z-z')^2]^{\frac{3}{2}}}$$

$$z - z' = z_1 \qquad dz\, dz' = dz_1 dz_2.$$
$$z = z_2$$

$$\vec{F} = -\hat{\rho} \frac{\mu_0}{4\pi} I_1 I_2\, a. \underbrace{\int_{-\infty}^{+\infty} dz_2}_{2L} \int_{-\infty}^{+\infty} dz_1 \frac{1}{(a^2 + z_1^2)^{3/2}}.$$

$$\frac{\vec{F}}{2L} = -\hat{\rho} \frac{\mu_0}{4\pi} I_1 I_2\, a \int_{-\infty}^{+\infty} dz_1 \frac{1}{(a^2 + z_1^2)^{3/2}}. \qquad t = \frac{z_1}{a}.$$

$\int_{-\infty}^{+\infty} dz_2 \leftarrow 2L$

$$= -\hat{\rho} \frac{\mu_0}{4\pi} \frac{I_1 I_2}{a} \int_{-\infty}^{+\infty} dt \frac{1}{(1+t^2)^{3/2}}$$

21 $$\int_{-\infty}^{+\infty} dt \, \frac{1}{(1+t^2)^{3/2}} = \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} d\theta \, \frac{\sec^2\theta}{\sec^3\theta}.$$

$t = \tan\theta$.
$dt = \sec^2\theta \, d\theta$.

$$= \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} d\theta \, \cos\theta = 2$$

22 Using 21 in 20 we have.

$$\frac{\vec{F}}{2L} = -\hat{\rho} \, \frac{\mu_0}{4\pi} \, \frac{2 I_1 I_2}{a}.$$

which matches the expression derived using magnetostatic energy.