SET – 29

Ampere's law

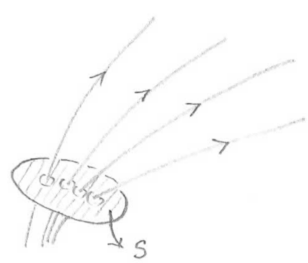
① Magnetostatics involves.

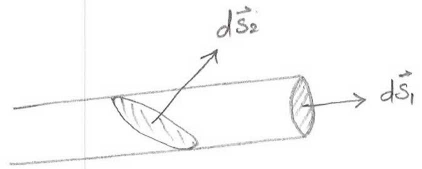
$$\vec{\nabla} \cdot \vec{B} = 0 \quad \Rightarrow \quad \vec{B} = \vec{\nabla} \times \vec{A}$$

$$\vec{\nabla} \times \vec{B} = \mu_0 \vec{J} \quad \Rightarrow \quad 0 = \vec{\nabla} \cdot \vec{J}$$

② Integrating over a surface (not necessarily closed)

$$\int_S d\vec{s} \cdot \vec{\nabla} \times \vec{B} = \mu_0 \int_S d\vec{s} \cdot \vec{J}$$





$$j = \frac{I}{\text{Area}}.$$

$$d\vec{S}_1 \cdot \vec{J} = d\vec{S}_2 \cdot \vec{J} = I$$

③ Using Stoke's theorem

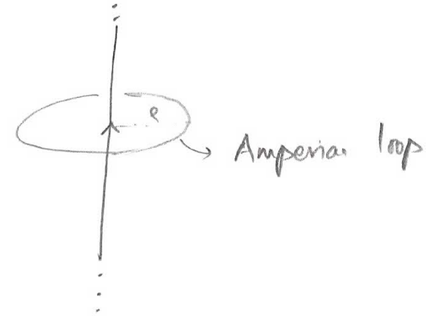
$$\int_S d\vec{s} \cdot \vec{\nabla} \times \vec{B} = \oint_{\substack{\text{enclosing}\\ \text{curve}}} d\vec{l} \cdot \vec{B}$$

④ Using ② and ③ we have the Ampere's law

$$\oint_{\substack{\text{enclosing}\\ \text{curve.}}} d\vec{l} \cdot \vec{B} = \mu_0 I$$

(5) Example I: Long (infinite wire)

$$\oint \vec{B} \cdot d\vec{l} = \mu_0 I$$

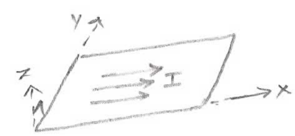


→ Conclude separately, using symmetry arguments, that $\vec{B}$ has only the $\hat{\phi}$ component.

$$B \, 2\pi\rho = \mu_0 I$$

$$\vec{B} = \frac{\mu_0}{4\pi} \frac{2I}{\rho} \hat{\phi}$$

(6) Example II: Infinite sheet

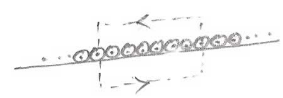


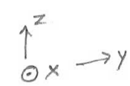
$$\oint d\vec{l} \cdot \vec{B} = \mu_0 I$$

$$K = \frac{I}{L} = \frac{\text{Current}}{\text{Length}}$$

→ Use symmetry to conclude.

$$\vec{B} = \begin{cases} \text{along } -\hat{y} & \text{above the sheet} \\ \text{along } +\hat{y} & \text{below the sheet} \end{cases}$$





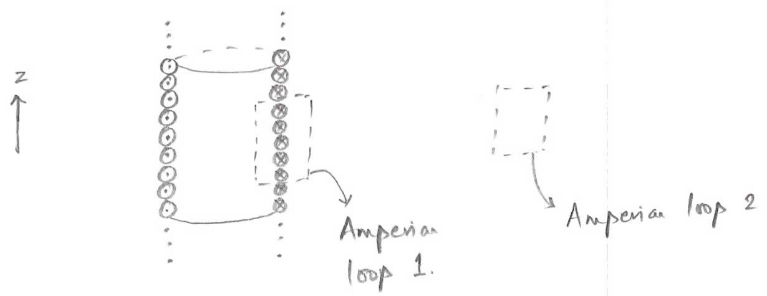
→ The sides of the Amperian loop that are perpendicular to the sheet, do not contribute, because $\vec{B} \cdot d\vec{l} = 0$.

$$BL + BL = \mu_0 K L$$

$$B = \frac{\mu_0 K}{2}$$

$$\vec{B} = \begin{cases} \frac{\mu_0}{4\pi} 2\pi K(-\hat{j}) & 0 < z \\ \frac{\mu_0}{4\pi} 2\pi K \hat{j} & z < 0 \end{cases}$$

⑦ Example III : Solenoid of infinite length



$n = \frac{\text{Turns}}{\text{Length}}$

I - current in the wire

→ Using symmetry argument conclude that $\vec{B}$ is along the $\hat{z}$ axis. Say along $\hat{z}$ inside and $-\hat{z}$ outside.

Loop 1 : $B_{in} L + B_{out} L = \mu_0 I n L$

$$B_{in} = \mu_0 I n + B_{out}$$

Loop 2 : $B_{out,a} L - B_{out,b} L = 0$

$B_{out,a}$ ↳ at radius ρ_a

$B_{out,b}$ ↳ at radius ρ_b.

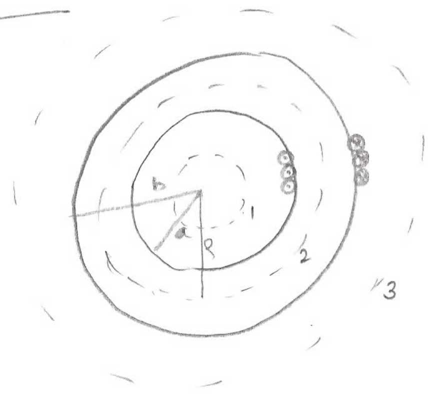
$$B_{out,a} = B_{out,b}$$

→ Require the magnetic field to be zero at ∞. Using $\rho_b = \infty$, we then conclude that $B_{out} = 0$.

↳ Also see the example of Toroid, which should limit to a solenoid.

$$\vec{B} = \begin{cases} \frac{\mu_0}{4\pi} 4\pi I n \, \hat{z} & \rho < a \\ 0 & a < \rho. \end{cases}$$

(8) Example IV : Toroid



→ Using symmetry conclude that $\vec{B}$ only has $\hat{\phi}$ component.

Loop 1 and 3 : $B = 0$ for $\rho < a$ and $b < \rho$.

$N = n_a \, 2\pi a = n_b \, 2\pi b$

Loop 2 : $B \, 2\pi\rho = \mu_0 I N$

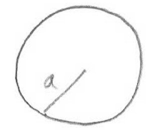
Thus,

$$\vec{B} = \begin{cases} \dfrac{\mu_0}{4\pi} \dfrac{2 I N}{\rho} \hat{\phi}, & a < \rho < b, \\ 0, & \text{otherwise}. \end{cases}$$

→ In the limit $a \to \infty$, $b \to \infty$, $b - a \to$ fixed, small θ, this is a solenoid. Thus, argue that $B_{out} = 0$ for a solenoid.

(9) Example V :

$$\vec{J}(\vec{r}) = \hat{z}\ C\ \rho^2\ \theta(a-\rho)$$



$\odot \hat{z}$

Let the total current be I. Then,

$$\int_0^{2\pi} d\phi \int_0^{\infty} \rho\, d\rho\ \hat{z} \cdot \vec{J}(\vec{r}) = I$$

$$\int_0^{2\pi} d\phi \int_0^{a} \rho\, d\rho\ C\ \rho^2 = I$$

$$2\pi C \frac{a^4}{4} = I \qquad \Rightarrow \qquad C = \frac{4I}{2\pi a^4}$$

$$\vec{J}(\vec{r}) = \hat{z}\ I\ \frac{4\rho^2}{2\pi a^4}\ \theta(a-\rho)$$

Outside : $a < \rho$:

$$B\ 2\pi\rho = \mu_0 I$$

$$B = \frac{\mu_0 I}{2\pi\rho}$$

Inside : $\rho < a$:

$$B\ 2\pi\rho = \mu_0 \int_0^{2\pi} d\phi \int_0^{\rho} \rho' d\rho'\, C\, \rho'^2$$

$$= \mu_0\ 2\pi C \frac{\rho^4}{4}$$

$$= \mu_0\ I\ \frac{\rho^4}{a^4}$$

$$\vec{B} = \begin{cases} \dfrac{\mu_0}{4\pi} \dfrac{2I}{\rho} \dfrac{\rho^4}{a^4}, & \rho < a, \\[2ex] \dfrac{\mu_0}{4\pi} \dfrac{2I}{\rho}, & a < \rho. \end{cases}$$

Homework:

do this for $n > 0$.

(10) Example VI :

$$\vec{J}(\vec{r}) = \hat{z}\, c\, \frac{e^{-\lambda\rho}}{\rho}$$

Let the total current be I. Thus,

$$\int_0^{2\pi} d\phi \int_0^{\infty} \rho\, d\rho\, \hat{z} \cdot \vec{J}(\vec{r}) = I$$

$$2\pi c \int_0^{\infty} \rho\, d\rho\, \frac{e^{-\lambda\rho}}{\rho} = I$$

$$2\pi \frac{c}{\lambda} = I \quad \Rightarrow \quad c = \frac{I\lambda}{2\pi}$$

$$\vec{J}(\vec{r}) = \hat{z}\, I\, \frac{\lambda}{2\pi} \frac{e^{-\lambda\rho}}{\rho}$$

$$B\, 2\pi\rho = \mu_0 \int_0^{2\pi} d\phi \int_0^{\rho} \rho' d\rho'\, c\, \frac{e^{-\lambda\rho'}}{\rho'}$$

$$= \mu_0\, 2\pi c\, \frac{1}{\lambda} \int_0^{\lambda\rho} dx\, e^{-x}$$

$$= \mu_0 I \left(1 - e^{-\lambda\rho}\right)$$

$$\vec{B} = \frac{\mu_0}{4\pi} \frac{2I}{\rho} \left[1 - e^{-\lambda\rho}\right] \hat{\phi}.$$