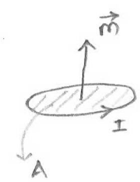
SET - 28

Magnetic dipole moment

1. An intuitive realization of a magnetic dipole moment is a (circular) current loop of area A, carrying a current I.



$\vec{m} = IA\,\hat{n}$ → normal to the loop.

2. But, a pure magnetic dipole moment (without having other multipoles) requires the $A \to 0$ and $I \to \infty$ such that IA is finite; which is called a point magnetic dipole moment.

3. In general, the magnetic dipole moment is defined as

$$\vec{m} = \frac{1}{2} q_a \vec{r}_a \times \vec{v}_a \;\longrightarrow\; \frac{1}{2}\int d^3r'\, \vec{r}\,' \times \vec{J}(\vec{r}\,')$$

4. Note that this is consistent with 1 because for a circular loop

$$\frac{1}{2} q\,|\vec{r} \times \vec{v}| = \frac{1}{2} q r v = \frac{qv}{2\pi r}\pi r^2$$

$$= \frac{q}{\Delta t}\underbrace{\frac{\Delta x}{2\pi r}}_{\text{no. of times the charges circles.}}\pi r^2$$

$$= IA.$$

(5) Let us now find the vector potential $\vec{A}$ and magnetic field $\vec{B}$ due to circular loop at distances very far from the loop, so that the loop can be approximated as a point dipole to the leading order.

(6)

$$\vec{J}(\vec{r}\,') = \hat{\phi}'\, I\, \delta(z'-0)\, \delta(\rho'-a)$$

$$\hat{\phi}' = -\sin\phi'\,\hat{i} + \cos\phi'\,\hat{j}$$

(7)

$$\vec{A}(\vec{r}) = \frac{\mu_0}{4\pi}\int d^3r'\, \frac{\vec{J}(\vec{r}\,')}{|\vec{r}-\vec{r}\,'|}$$

$$= \frac{\mu_0}{4\pi}\int_0^{2\pi} d\phi' \int_0^{\infty}\rho' d\rho' \int_{-\infty}^{+\infty} dz'\; \frac{I\,\delta(z'-0)\,\delta(\rho'-a)\left[-\sin\phi'\,\hat{i} + \cos\phi'\,\hat{j}\right]}{\sqrt{(z-z')^2 + \rho^2 + \rho'^2 - 2\rho\rho'\cos(\phi-\phi')}}$$

$$= \frac{\mu_0}{4\pi}\, I a \int_0^{2\pi} d\phi'\; \frac{\left[-\sin\phi'\,\hat{i} + \cos\phi'\,\hat{j}\right]}{\sqrt{\underbrace{z^2+\rho^2}_{r^2} + a^2 - 2a\rho\cos(\phi-\phi')}}$$

$$= \frac{\mu_0}{4\pi}\, \frac{Ia}{r} \int_0^{2\pi} d\phi'\; \frac{\left[-\sin\phi'\,\hat{i} + \cos\phi'\,\hat{j}\right]}{\sqrt{1 - \frac{2a\rho}{r^2}\cos(\phi-\phi') + \frac{a^2}{r^2}}}$$

$$= \frac{\mu_0}{4\pi}\, \frac{Ia}{r} \int_0^{2\pi} d\phi' \left[-\sin\phi'\,\hat{i} + \cos\phi'\,\hat{j}\right]\left[\underbrace{1}_{=0 \text{ because } \phi' \text{ integral} \to 0} + \frac{a\rho}{r^2}\cos(\phi-\phi') + O\left(\frac{a}{r}\right)^2\right]$$

$$= \frac{\mu_0}{4\pi}\, \frac{I a^2 \rho}{r^3} \int_0^{2\pi} d\phi' \left[-\sin\phi'\,\hat{i} + \cos\phi'\,\hat{j}\right] \cos(\phi-\phi') + \frac{1}{r} O\left(\frac{a}{r}\right)^3$$

8 $$\int_0^{2\pi} d\phi' \left[-\sin\phi' \,\hat{i} + \cos\phi' \,\hat{j} \right] \cos(\phi - \phi')$$

$$\int_0^{2\pi} d\phi' \cos\phi' \sin\phi' = 0$$

$$= \int_0^{2\pi} d\phi' \left[-\sin\phi' \,\hat{i} + \cos\phi' \,\hat{j} \right] \left(\cos\phi \cos\phi' + \sin\phi \sin\phi' \right)$$

$$= -\sin\phi \,\hat{i} \underbrace{\int_0^{2\pi} d\phi' \sin^2\phi'}_{\frac{1}{2}\times 2\pi} + \cos\phi \,\hat{j} \underbrace{\int_0^{2\pi} d\phi' \cos^2\phi'}_{\frac{1}{2}\times 2\pi}$$

$$= \frac{1}{2}\left[-\sin\phi \,\hat{i} + \cos\phi \,\hat{j} \right] \times 2\pi$$

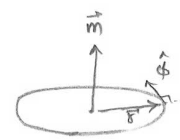
$$= \pi \,\hat{\phi}$$

9 Using 8 in 7

$$\vec{A}(\vec{r}) = \frac{\mu_0}{4\pi} \frac{I a^2 \rho}{r^3} \pi \,\hat{\phi} + O\left(\frac{a^3}{r^3}\right)$$

$$= \frac{\mu_0}{4\pi} \frac{m \, r \sin\theta}{r^3} \hat{\phi} + O\left(\frac{a^3}{r^3}\right)$$

$$= \frac{\mu_0}{4\pi} \frac{\vec{m} \times \vec{r}}{r^3} + O\left(\frac{a^3}{r^3}\right)$$



10 $$\vec{B}(\vec{r}) = \vec{\nabla} \times \vec{A}$$

$$= \frac{\mu_0}{4\pi} \vec{\nabla} \times \left(\vec{m} \times \frac{\vec{r}}{r^3} \right)$$

$$= \frac{\mu_0}{4\pi} \vec{m} \,\vec{\nabla} \cdot \frac{\vec{r}}{r^3} - \frac{\mu_0}{4\pi} \vec{m} \cdot \vec{\nabla} \,\frac{\vec{r}}{r^3}$$

(11) $$\vec{\nabla} \cdot \left(\frac{\vec{r}}{r^3} \right) = \frac{3}{r^3} - 3 \frac{1}{r^3} \longrightarrow 4\pi\, \delta^{(3)}(\vec{r})$$

(12) Thus,

$$\vec{\nabla} \frac{1}{r} = - \frac{\vec{r}}{r^3}$$

$$\vec{A}(\vec{r}) = \frac{\mu_0}{4\pi} \frac{\vec{m} \times \vec{r}}{r^3}$$

$$\vec{B}(\vec{r}) = \frac{\mu_0}{4\pi} 4\pi\, \vec{m}\, \delta^{(3)}(\vec{r}) - \frac{\mu_0}{4\pi} \vec{m} \cdot \vec{\nabla} \frac{\vec{r}}{r^3}$$

$$= \frac{\mu_0}{4\pi} 4\pi\, \vec{m}\, \delta^{(3)}(\vec{r}) + \frac{\mu_0}{4\pi} (\vec{m} \cdot \vec{\nabla})\, \vec{\nabla} \frac{1}{r}$$

(13) Checks:

(i) $\vec{\nabla} \cdot \vec{A} = 0$ (Choice of gauge.)

(ii) $\vec{\nabla} \cdot \vec{B} = 0$ (No monopoles.)

(14) $$\vec{\nabla} \cdot \vec{A} = \frac{\mu_0}{4\pi} \vec{\nabla} \cdot \vec{m} \times \frac{\vec{r}}{r^3}$$

$$= - \frac{\mu_0}{4\pi} \vec{m} \cdot \vec{\nabla} \times \frac{\vec{r}}{r^3}$$

$$= 0. \quad \checkmark$$

(15) $$\vec{\nabla} \cdot \vec{B} = \frac{\mu_0}{4\pi} 4\pi\, \vec{m} \cdot \vec{\nabla}\, \delta^{(3)}(\vec{r}) + \frac{\mu_0}{4\pi} \vec{m} \cdot \vec{\nabla}\, \nabla^2 \frac{1}{r}$$

$$= \frac{\mu_0}{4\pi} 4\pi\, (\vec{m} \cdot \vec{\nabla})\, \delta^{(3)}(\vec{r}) - \frac{\mu_0}{4\pi} (\vec{m} \cdot \vec{\nabla})\, 4\pi\, \delta^{(3)}(\vec{r})$$

$$= 0 \quad \checkmark$$

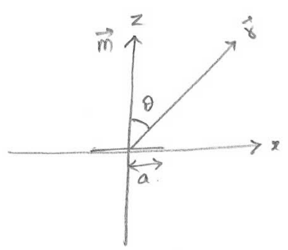
(16) Let us evaluate $\vec{B}$ for $r \neq 0$. Using (12)

$$\vec{B}(\vec{r}) = -\frac{\mu_0}{4\pi}\, \vec{m} \cdot \vec{\nabla}\, \frac{\vec{r}}{r^3} \qquad (r \neq 0)$$

$$= -\frac{\mu_0}{4\pi}\, \vec{m} \cdot \left[\frac{\overleftrightarrow{1}}{r^3} - \frac{3\,\vec{r}\,\vec{r}}{r^5} \right]$$

$$= \frac{\mu_0}{4\pi}\, \frac{1}{r^5} \left[3\,(\vec{m}\cdot\vec{r})\,\vec{r} - r^2\,\vec{m} \right]$$

(17)

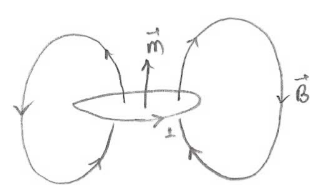


$$\vec{B}(\vec{r}) = \frac{\mu_0}{4\pi}\, \frac{1}{r^5} \left[\vec{r}\; 3 m r \cos\theta - \vec{m}\, r^2 \right]$$

$\theta = 0$: $$\vec{B}(\vec{r}) = \frac{\mu_0}{4\pi}\, \frac{1}{r^5} \left[\hat{z}\; 3 m r^2 - m r^2\, \hat{z} \right] = \frac{\mu_0}{4\pi}\, \frac{2m}{r^3}\, \hat{z}$$

$\theta = \frac{\pi}{2}$: $$\vec{B}(\vec{r}) = \frac{\mu_0}{4\pi}\, \frac{1}{r^5} \left[0 - \hat{z}\, m r^2 \right] = -\frac{\mu_0}{4\pi}\, \frac{m}{r^3}\, \hat{z}$$

$\theta = \pi$: $$\vec{B}(\vec{r}) = \frac{\mu_0}{4\pi}\, \frac{1}{r^5} \left[+\hat{z}\; 3 m r^2 - m r^2\, \hat{z} \right] = \frac{\mu_0}{4\pi}\, \frac{2m}{r^3}\, \hat{z}$$



(18) Multipole expansion

Multipole expansion of

$$\vec{A}(\vec{r}) = \frac{\mu_0}{4\pi} \int d^3r' \frac{\vec{J}(\vec{r}')}{|\vec{r} - \vec{r}'|}$$

should lead to the same result.

(19) $$\frac{1}{|\vec{r} - \vec{r}'|} = \frac{1}{r} + \frac{\vec{r}\cdot\vec{r}'}{r^3} + \cdots$$

(20) $$\vec{A}(\vec{r}) = \frac{\mu_0}{4\pi} \int d^3r' \, \vec{J}(\vec{r}') \left[\frac{1}{r} + \frac{\vec{r}\cdot\vec{r}'}{r^3} + \cdots \right]$$

$$= \frac{\mu_0}{4\pi} \frac{1}{r} \int d^3r' \, \vec{J}(\vec{r}') + \frac{\mu_0}{4\pi} \frac{\vec{r}}{r^3} \cdot \int d^3r' \, \vec{r}' \, \vec{J}(\vec{r}') + \cdots$$

(21) We know

$$\vec{\nabla} \cdot \vec{J} = 0$$

Thus, we can show that

$$0 = \int d^3r \, \vec{r} \, \vec{\nabla}\cdot\vec{J}$$

$$= -\int d^3r \, \vec{J} \cdot (\vec{\nabla}\,\vec{r}) + \underbrace{\int d^3r \, \vec{\nabla}\cdot(\vec{J}\,\vec{r})}_{=0}$$

$=0$ for confined current distributions.

$$= -\int d^3r \, \vec{J}(\vec{r})$$

22 Using 21 in 20 the first term in multipole expansion contribute zero, which is the statement of no magnetic monopole.

23 Similarly, we have

$$0 = \int d^3r \; \vec{r}\,\vec{r}\; \vec{\nabla}\cdot\vec{J}$$

$$= -\int d^3r \; \vec{J}\cdot\vec{\nabla}(\vec{r}\,\vec{r}) \quad + \underbrace{\text{surface term}}_{=0}$$

$$= -\int d^3r \; \vec{J}_i \cdot \left(\overleftrightarrow{I}_{im}\,\vec{r}_n + \vec{r}_m\,\overleftrightarrow{I}_{in} \right)$$

$$= -\int d^3r \left(\vec{J}\,\vec{r} + \vec{r}\,\vec{J} \right)$$

24 Using 23 and 22 in 20

$$\vec{A}(\vec{r}) = 0 + \frac{\mu_0}{4\pi}\frac{\vec{r}}{r^3}\cdot\frac{1}{2}\int d^3r' \Big[\underbrace{\vec{r}'\,\vec{J} + \vec{J}\,\vec{r}'}_{\substack{=0 \\ \text{using } 23}} + \vec{r}'\,\vec{J} - \vec{J}\,\vec{r}' \Big]$$

$$= \frac{\mu_0}{4\pi}\,\frac{1}{2}\,\frac{1}{r^3}\int d^3r' \left(\vec{r}\cdot\vec{r}'\,\vec{J} - \vec{r}\cdot\vec{J}\,\vec{r}' \right) \qquad \vec{J} = \vec{J}(\vec{r}')$$

$$= \frac{\mu_0}{4\pi}\,\frac{1}{2}\,\frac{1}{r^3}\int d^3r' \; \vec{r}\times\left(\vec{J}\times\vec{r}'\right)$$

$$= \frac{\mu_0}{4\pi}\,\frac{1}{2}\left[\int d^3r' \; \vec{r}'\times\vec{J}(\vec{r}')\right]\times\frac{\vec{r}}{r^3}$$

$$\vec{m} = \frac{1}{2}\int d^3r' \; \vec{r}'\times\vec{J}(\vec{r}')$$

$$= \frac{\mu_0}{4\pi}\,\frac{\vec{m}\times\vec{r}}{r^3}$$