SET : 27

Magnetostatics

(1) Maxwell's equations

$$\vec{\nabla} \cdot (\epsilon_0 \vec{E}) = \rho$$

$$-\vec{\nabla} \times \vec{E} = \frac{\partial \vec{B}}{\partial t}$$

$$\vec{\nabla} \cdot \vec{B} = 0$$

$$\vec{\nabla} \times \left(\frac{1}{\mu_0} \vec{B}\right) = \frac{\partial}{\partial t}(\epsilon_0 \vec{E}) + \vec{J}$$

$$\epsilon_0 \vec{E} \rightarrow \vec{D} = \epsilon_0 \vec{E} + \vec{P}$$

$$\frac{1}{\mu_0} \vec{B} \rightarrow \vec{H} = \frac{1}{\mu_0} \vec{B} - \vec{M}$$

$$\frac{\partial \rho}{\partial t} + \vec{\nabla} \cdot \vec{J} = 0$$

$$\vec{F} = q[\vec{E} + \vec{v} \times \vec{B}]$$

$$\mu_0 = 4\pi \times 10^{-7} \ \frac{\text{Tesla-meter}}{\text{Ampere}}$$

(2) Statics

$$\frac{\partial \rho}{\partial t} = 0 \qquad \frac{\partial \vec{J}}{\partial t} = 0 \qquad \frac{\partial \vec{E}}{\partial t} = 0 \qquad \frac{\partial \vec{B}}{\partial t} = 0$$

For consistency we also need

$$\underbrace{\cancel{\frac{\partial \rho}{\partial t}}}_{=0} + \vec{\nabla} \cdot \vec{J} = 0$$

(3) Electrostatics

$$\vec{\nabla} \cdot (\epsilon_0 \vec{E}) = \rho$$

$$\vec{\nabla} \times \vec{E} = 0$$

Magnetostatics

$$\vec{\nabla} \cdot \vec{B} = 0 \quad \Rightarrow \quad \vec{B} = \vec{\nabla} \times \vec{A}$$

$$\vec{\nabla} \times \left(\frac{1}{\mu_0} \vec{B}\right) = \vec{J} \quad \Rightarrow \quad \vec{\nabla} \cdot \vec{J} = 0$$

(4) Thus, the fundamental equation in magnetostatics is

$$\vec{\nabla} \times \left(\frac{1}{\mu_0} \vec{\nabla} \times \vec{A} \right) = \vec{J}$$

$$\vec{\nabla} \times (\vec{\nabla} \times \vec{A}) = \mu_0 \vec{J} \qquad \text{(in vacuum)}$$

$$\vec{\nabla} (\vec{\nabla} \cdot \vec{A}) - \nabla^2 \vec{A} = \mu_0 \vec{J}$$

(5) <u>Gauge transformations</u>

→ Maxwell's equations (not only statics) are invariant under

$$\vec{A} \to \vec{A} + \vec{\nabla} \lambda \qquad \lambda = \lambda(\vec{r}, t) \to \text{arbitrary}$$

$$\phi \to \phi - \frac{\partial}{\partial t} \lambda$$

→ $$\vec{E} \to -\vec{\nabla}\left(\phi - \frac{\partial}{\partial t}\lambda\right) - \frac{\partial}{\partial t}\left(\vec{A} + \vec{\nabla}\lambda\right) = \vec{E}$$

$$\vec{B} \to \vec{\nabla} \times (\vec{A} + \vec{\nabla}\lambda) = \vec{B}$$

→ Radiation (Coulomb) gauge : $\vec{\nabla} \cdot \vec{A} = 0$.

(6) In the radiation gauge (4) becomes.

$$-\nabla^2 \vec{A} = \mu_0 \vec{J} \qquad -\nabla^2 G = 1, \quad G = (-\nabla^2)^{-1} = \frac{1}{4\pi r}$$

$$\vec{A}(\vec{r}) = \mu_0 \int d^3 r' \, G(\vec{r}, \vec{r}') \, \vec{J}(\vec{r}')$$

$$= \frac{\mu_0}{4\pi} \int d^3 r' \, \frac{\vec{J}(\vec{r}')}{|\vec{r} - \vec{r}'|}$$

→ direction of $\vec{A}$ due to line segment is decided by $\vec{J}$.

(7) Consistency check for radiation gauge:

(i) If $\vec{\nabla}\cdot\vec{A} \neq 0$

choose $\vec{A}' = \vec{A} + \vec{\nabla}\lambda$

then $\vec{\nabla}\cdot\vec{A}' = 0$ requires

$$-\nabla^2\lambda = \vec{\nabla}\cdot\vec{A}$$

which exists. In principle λ is determined by solving the Laplacian with $\vec{\nabla}\cdot\vec{A}$ as the source.

$$\frac{\partial}{\partial x} f(x-y) = -\frac{\partial}{\partial y} f(x-y)$$

(ii)
$$\vec{\nabla}\cdot\vec{A} = \frac{\mu_0}{4\pi}\,\vec{\nabla}\cdot\int d^3r'\,\frac{\vec{J}(\vec{r}')}{|\vec{r}-\vec{r}'|}$$

$$= \frac{\mu_0}{4\pi}\int d^3r'\,\vec{J}(\vec{r}')\cdot\vec{\nabla}\frac{1}{|\vec{r}-\vec{r}'|}$$

$$= +\frac{\mu_0}{4\pi}\int d^3r'\,\frac{(\vec{\nabla}\cdot\vec{J})}{|\vec{r}-\vec{r}'|} - \underbrace{\frac{\mu_0}{4\pi}\int d^3r'\,\vec{\nabla}\cdot\left(\frac{\vec{J}(\vec{r}')}{|\vec{r}-\vec{r}'|}\right)}_{=0}$$

($\because$ Green's fn. goes to zero)

($\vec{\nabla}\cdot\vec{J} = 0$ in statics.)

$$= 0$$

(8) The magnetic field is determined as.

$$\vec{B}(\vec{r}) = \vec{\nabla}\times\vec{A}$$

$$= -\frac{\mu_0}{4\pi}\int d^3r'\,\vec{J}(\vec{r}')\times\vec{\nabla}\frac{1}{|\vec{r}-\vec{r}'|}$$

$$= \frac{\mu_0}{4\pi}\int d^3r'\,\vec{J}(\vec{r}')\times\frac{\vec{r}-\vec{r}'}{|\vec{r}-\vec{r}'|^3}.$$

(8A) Force on another wire

$$\vec{F} = q\,\vec{v}\times\vec{B} \rightarrow \int d^3r'\,\vec{J}_2(\vec{r}')\times\vec{B}(\vec{r}')$$

9 I leave it as an exercise to verify

$$\vec{\nabla} \cdot \vec{B} = \vec{\nabla} \cdot \frac{\mu_0}{4\pi} \int d^3r' \, \vec{J}(\vec{r}') \times \frac{\vec{r} - \vec{r}'}{|\vec{r} - \vec{r}'|^3} = 0$$

$$\vec{\nabla} \times \vec{B} = \vec{\nabla} \times \frac{\mu_0}{4\pi} \int d^3r' \, \vec{J}(\vec{r}') \times \frac{\vec{r} - \vec{r}'}{|\vec{r} - \vec{r}'|^3} = \mu_0 \vec{J}(\vec{r})$$

10 Example I

Consider a point charge $\qquad \vec{V}_a = \frac{d\vec{r}_a}{dt}$

$$\rho(\vec{r}) = q \, \delta^{(3)}(\vec{r} - \vec{r}_a)$$

$$\vec{J}(\vec{r}) = q \, \vec{V}_a \, \delta^{(3)}(\vec{r} - \vec{r}_a)$$

Then,

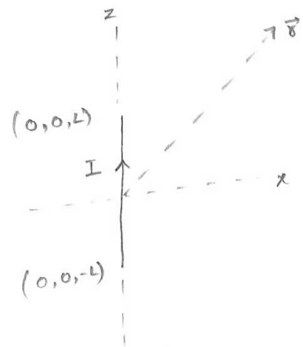
$$\vec{A}(\vec{r}) = \frac{\mu_0}{4\pi} \int d^3r' \, \frac{q \vec{V}_a \, \delta^{(3)}(\vec{r}' - \vec{r}_a)}{|\vec{r} - \vec{r}'|}$$

$$= \frac{\mu_0}{4\pi} \frac{q \vec{V}_a}{|\vec{r} - \vec{r}_a|}$$

$$\vec{B}(\vec{r}) = \frac{\mu_0}{4\pi} \int d^3r' \, q \vec{V}_a \, \delta^{(3)}(\vec{r}' - \vec{r}_a) \times \frac{\vec{r} - \vec{r}'}{|\vec{r} - \vec{r}'|^3}$$

$$= \frac{\mu_0}{4\pi} q \vec{V}_a \times \frac{\vec{r} - \vec{r}_a}{|\vec{r} - \vec{r}_a|^3}$$

(11) Example II : Current carrying line segment



$$\vec{J}(\vec{r}') = \hat{z}\, I\, \delta(x'-0)\, \delta(y'-0) \quad \theta(-L<z<L)$$

(12)
$$\vec{A}(\vec{r}) = \frac{\mu_0}{4\pi} \int d^3r' \, \frac{\vec{J}(\vec{r}')}{|\vec{r}-\vec{r}'|}$$

$$= \frac{\mu_0}{4\pi} \int_{-L}^{L} dz' \int_{-\infty}^{+\infty} dx' \int_{-\infty}^{+\infty} dy' \, \frac{\hat{z}\, I\, \delta(x'-0)\, \delta(y'-0)}{\sqrt{(x-x')^2 + (y-y')^2 + (z-z')^2}}$$

$$\vec{A} = (0, 0, A_z)$$

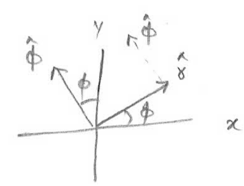
$$= \frac{\mu_0}{4\pi}\, \hat{z}\, I \int_{-L}^{L} dz' \, \frac{1}{\sqrt{x^2+y^2+(z-z')^2}}$$

$$\begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ 0 & 0 & A_z \end{vmatrix}$$

(13)
$$\vec{B}(\vec{r}) = \vec{\nabla} \times \vec{A}$$

$$= \hat{i}\, \frac{\partial A_z}{\partial y} - \hat{j}\, \frac{\partial}{\partial x} A_z$$

$$= \frac{\mu_0}{4\pi}\, I \int_{-L}^{+L} dz' \, \frac{\left[-y\, \hat{i} + x\, \hat{j} \right]}{\left[x^2 + y^2 + (z-z')^2 \right]^{\frac{3}{2}}}$$



(19)
$$-y\, \hat{i} + x\, \hat{j} = \left[-\sin\phi\, \hat{i} + \cos\phi\, \hat{j} \right] \rho$$

$$= \rho\, \hat{\phi}$$

(20)
$$\vec{B}(\vec{r}) = \hat{\phi}\, \frac{\mu_0 I}{4\pi}\, \rho \int_{-L}^{L} dz' \, \frac{1}{\left[x^2+y^2+(z-z')^2 \right]^{\frac{3}{2}}}$$

$$x^2+y^2=\rho^2$$

$$z-z' = \rho \tan\theta .$$

(21) $$\int_{-L}^{L} dz' \frac{1}{\left[x^2+y^2+(z-z')^2\right]^{3/2}} = \frac{1}{\rho^2}\int_{\tan^{-1}\frac{z-L}{\rho}}^{\tan^{-1}\frac{z+L}{\rho}} \cos\theta \, d\theta$$

$$= \frac{1}{\rho^2} \sin\theta \Big|_{\tan^{-1}\frac{z-L}{\rho}}^{\tan^{-1}\frac{z+L}{\rho}}$$

$$\sin\theta = \tan\theta\cos\theta = \frac{\tan\theta}{\sqrt{1+\tan^2\theta}}$$

$$= \frac{1}{\rho^2}\left[\frac{\frac{z+L}{\rho}}{\sqrt{1+\left(\frac{z+L}{\rho}\right)^2}} - \frac{\frac{z-L}{\rho}}{\sqrt{1+\left(\frac{z-L}{\rho}\right)^2}}\right]$$

$$= \frac{1}{\rho^2}\left[\frac{z+L}{\sqrt{\rho^2+(z+L)^2}} - \frac{(z-L)}{\sqrt{\rho^2+(z-L)^2}}\right]$$

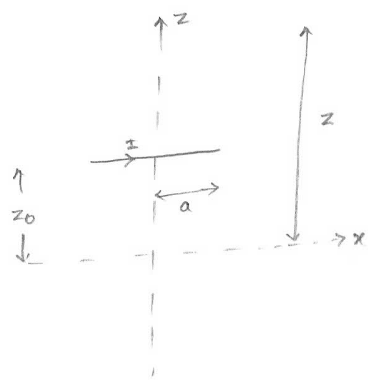
(22) Using (21) in (20)

$$\vec{B}(\vec{r}) = \hat{\phi}\frac{\mu_0 I}{4\pi\rho}\left[\frac{z+L}{\sqrt{\rho^2+(z+L)^2}} - \frac{(z-L)}{\sqrt{\rho^2+(z-L)^2}}\right]$$

(23) $L \to \infty$: $$\vec{B}(\vec{r}) = \hat{\phi}\frac{\mu_0 I}{4\pi\rho}\, 2$$

$z = 0$: $$\vec{B}(\vec{r}) = \hat{\phi}\frac{\mu_0 I}{4\pi\rho}\frac{2L}{\sqrt{\rho^2+L^2}}$$

(24) Example III : A finite solenoid.



$$n = \frac{\text{no. of turns}}{\text{length}}$$

$$I_{dz_0} = I\, n\, dz_0$$

$$\vec{J}(\vec{r}') = I_{dz_0}\, \delta(z'-z_0)\, \delta(\rho'-a)\, \hat{\phi}'$$

$$= I\, n\, dz_0\, \delta(z'-z_0)\, \delta(\rho'-a) \left[-\sin\phi'\, \hat{i} + \cos\phi'\, \hat{j}\right]$$

(25) $$\vec{A}(\vec{r}) = \frac{\mu_0}{4\pi} \int d^3r' \frac{\vec{J}(\vec{r}')}{|\vec{r}-\vec{r}'|}$$

$$= \frac{\mu_0}{4\pi} \int_{-\infty}^{+\infty} dz' \int_0^{\infty} \rho' d\rho' \int_0^{2\pi} d\phi' \frac{I\, n\, dz_0 \left[-\sin\phi'\, \hat{i} + \cos\phi'\, \hat{j}\right] \delta(z'-z_0)\, \delta(\rho'-a)}{\sqrt{(z-z')^2 + \rho^2 + \rho'^2 - 2\rho\rho' \cos(\phi-\phi')}}$$

$$= \frac{\mu_0 I n}{4\pi}\, dz_0\, a \int_0^{2\pi} d\phi' \frac{\left[-\sin\phi'\, \hat{i} + \cos\phi'\, \hat{j}\right]}{\sqrt{(z-z_0)^2 + \rho^2 + a^2 - 2a\rho\cos(\phi-\phi')}}$$

(26) $$\vec{A}(\underbrace{0,0}_{\rho=0},z) = \frac{\mu_0 I n}{4\pi}\, dz_0\, a \int_0^{2\pi} d\phi' \frac{\left[-\sin\phi'\, \hat{i} + \cos\phi'\, \hat{j}\right]}{\sqrt{(z-z_0)^2 + a^2}}$$

$$= 0$$

27 <u>Caution</u>:

$$\vec{A}(0,0,z) = 0 \quad \not\Rightarrow \quad \vec{B}(0,0,z) = \vec{\nabla} \times \vec{A}$$

28
$$\vec{B} = \vec{\nabla} \times \vec{A}$$
$$= \frac{\mu_0}{4\pi} \int d^3 r' \; \vec{J}(\vec{r}') \times \frac{\vec{r} - \vec{r}'}{|\vec{r} - \vec{r}'|^3}$$
$$= \frac{\mu_0}{4\pi} \int_{-\infty}^{+\infty} dz' \int_0^{\infty} \rho' d\rho' \int_0^{2\pi} d\phi' \quad I n \, dz_0 \; \delta(z' - z_0) \; \delta(\rho' - a)$$
$$\times \frac{\left[-\sin\phi' \, \hat{i} + \cos\phi' \, \hat{j} \right] \times \left[(x - x') \, \hat{i} + (y - y') \, \hat{j} + (z - z') \, \hat{k} \right]}{\left[\sqrt{(z - z')^2 + \rho^2 + \rho'^2 - 2\rho\rho' \cos(\phi - \phi')} \right]^3}$$

29
$$\vec{B}(0,0,z) = \frac{\mu_0 I n}{4\pi} \, dz_0 \; a \int_0^{2\pi} d\phi' \; \frac{\left[-\sin\phi' \, \hat{i} + \cos\phi' \, \hat{j} \right] \times \left[-a\cos\phi' \, \hat{i} - a\sin\phi' \, \hat{j} + (z - z_0) \, \hat{k} \right]}{\left[\sqrt{(z - z_0)^2 + a^2} \right]^3}$$

$$= \frac{\mu_0 I n}{4\pi} \, dz_0 \; \frac{a}{\left[a^2 + (z - z_0)^2 \right]^{\frac{3}{2}}} \int_0^{2\pi} d\phi' \; a \, \hat{k}$$

$$= \frac{\mu_0 I n}{4\pi} \, dz_0 \; \frac{2\pi a^2}{\left[a^2 + (z - z_0)^2 \right]^{\frac{3}{2}}} \; \hat{k}$$

$$= \frac{\mu_0 I n}{2} \; \hat{k} \; \frac{a^2 \, dz_0}{\left[a^2 + (z - z_0)^2 \right]^{\frac{3}{2}}}$$

30 $$\vec{B}_{solenoid}(0,0,z) = \int_{-L}^{L} dz_0 \quad \vec{B}(0,0,z)$$

$$= \frac{\mu_0 I n}{2} \hat{k} \int_{-L}^{L} dz_0 \frac{a^2}{\left[a^2+(z-z_0)^2\right]^{\frac{3}{2}}}$$

$$= \frac{\mu_0 I n}{2} \hat{k} \left[\frac{z+L}{\sqrt{a^2+(z+L)^2}} - \frac{z-L}{\sqrt{a^2+(z-L)^2}} \right]$$

31 $\underline{L \to \infty}$:

$$\vec{B}_{solenoid}(0,0,z) = \mu_0 I n \hat{k}$$

$\underline{\underline{z=0}}$:

$$\vec{B}_{solenoid}(0,0,0) = \frac{\mu_0 I n}{2} \hat{k} \frac{2L}{\sqrt{a^2+L^2}}$$