SET-26

A simple application of spherical harmonics

1. We list the most useful identities for spherical harmonics here:

(i) $$Y_{lm}(\theta,\phi) = \sum_{m=-l}^{l} \sqrt{\frac{2l+1}{4\pi}} \sqrt{\frac{(l+m)!}{(l-m)!}} \left(\frac{e^{i\phi}}{\sin\theta}\right)^m \left(\frac{d}{dt}\right)^{l-m} \frac{(t^2-1)^l}{2^l\, l!} \Bigg|_{t=\cos\theta}.$$

(ii) $$\int d\Omega \; Y_{lm}(\theta,\phi)\, Y^*_{l'm'}(\theta,\phi) = \delta_{ll'}\,\delta_{mm'}$$

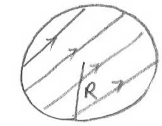
(iii) $$P_l(\cos\theta) = \sqrt{\frac{4\pi}{2l+1}}\; Y_{l0}(\theta,\phi)$$

(iv) $$\frac{1}{|\vec{r}-\vec{r}\,'|} = \sum_{l=0}^{\infty}\sum_{m=-l}^{l} \frac{r_<^l}{r_>^{l+1}}\,\frac{4\pi}{2l+1}\, Y_{lm}(\theta,\phi)\, Y^*_{lm}(\theta',\phi')$$

$$\xrightarrow{\theta'=0} \sum_{l=0}^{\infty} \frac{r_<^l}{r_>^{l+1}}\, P_l(\cos\theta)$$

(v) $$P_l(\cos\gamma) = \frac{4\pi}{2l+1}\sum_{m=-l}^{l} Y_{lm}(\theta,\phi)\, Y^*_{lm}(\theta',\phi')$$

(2) Let us recollect a calculation we did a while ago. Consider a sphere with polarizability $\vec{P}$



$$\vec{P} = \vec{C}\;\theta(R-r)$$

$$\rho_{eff} = -\vec{\nabla}\cdot\vec{P} = -\vec{C}\cdot\vec{\nabla}\,\theta(R-r)$$
$$= (\vec{C}\cdot\hat{r})\;\delta(r-R)$$

(3)
$$\vec{\nabla}\cdot\epsilon_0\vec{E} = \underbrace{\rho(\vec{r})}_{=0} - \vec{\nabla}\cdot\vec{P}$$

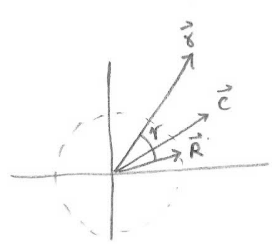
$$-\epsilon_0\nabla^2\phi(\vec{r}) = (\vec{C}\cdot\hat{r})\;\delta(r-R)$$

$$\phi(\vec{r}) = \frac{1}{4\pi\epsilon_0}\int d^3r'\,\frac{\rho_{eff}(\vec{r}')}{|\vec{r}-\vec{r}'|}$$

$$= \frac{1}{4\pi\epsilon_0}\int_0^\infty r'^2 dr'\int_0^{2\pi}d\phi'\int_0^\pi \sin\theta'\,d\theta'\;\frac{(\vec{C}\cdot\hat{r}')\;\delta(r'-R)}{\sqrt{r^2+r'^2-2rr'\cos\gamma}}$$

$$= \frac{R^2}{4\pi\epsilon_0}\int_0^{2\pi}d\phi'\int_0^\pi \sin\theta'\,d\theta'\;\frac{(\vec{C}\cdot\hat{R})}{\sqrt{r^2+R^2-2rR\cos\gamma}}$$

(4) Three vector : $\vec{C}$, $\vec{r}$, $\vec{R}$



$$\vec{r} = r(\sin\theta\cos\phi,\ \sin\theta\sin\phi,\ \cos\theta)$$
$$\vec{R} = R(\sin\theta'\cos\phi',\ \sin\theta'\sin\phi',\ \cos\theta')$$

$$\cos\gamma = \cos\theta\cos\theta' + \sin\theta\sin\theta'\cos(\phi-\phi')$$

5 Earlier, we chose $\vec{r}$ along the z-axis.

$$\phi(\vec{r}) = \frac{R^2}{4\pi\epsilon_0} \int_0^{2\pi} d\phi' \int_0^{\pi} \sin\theta' d\theta' \frac{\vec{C}\cdot(\sin\theta'\cos\phi'\,\hat{i} + \sin\theta'\sin\phi'\,\hat{j} + \cos\theta'\,\hat{k})}{\sqrt{r^2 + R^2 - 2rR\cos\theta'}}$$

$$= \frac{(\vec{C}\cdot\hat{k})}{4\pi\epsilon_0} 2\pi R^2 \int_0^{\pi} \sin\theta' d\theta \frac{\cos\theta'}{\sqrt{r^2 + R^2 - 2rR\cos\theta'}}$$

$$= \frac{\vec{C}\cdot\hat{k}}{4\pi\epsilon_0} 2\pi R^2 \frac{2}{3} \frac{r_<}{r_>^2}$$

$r_< = \mathrm{Min}(r, R)$

$r_> = \mathrm{Max}(r, R)$

$$\phi(\vec{r}) = \frac{\vec{C}\cdot\hat{r}}{4\pi\epsilon_0} \frac{4\pi}{3} R^2 \frac{r_<}{r_>^2}$$

6 Let us now choose $\vec{C}$ along the z-axis.

$$\phi(\vec{r}) = \frac{R^2}{4\pi\epsilon_0} \int_0^{2\pi} d\phi' \int_0^{\pi} \sin\theta' d\theta' \frac{C\cos\theta'}{\sqrt{r^2 + R^2 - 2rR\cos\gamma}}$$

7
$$P_1(\cos\theta') = \cos\theta' = \sqrt{\frac{4\pi}{3}}\, Y_{10}(\theta', \phi')$$

$$\frac{1}{\sqrt{r^2 + R^2 - 2rR\cos\gamma}} = \sum_{l=0}^{\infty} \sum_{m=-l}^{l} \frac{r_<^l}{r_>^{l+1}} \frac{4\pi}{2l+1} Y_{lm}(\theta, \phi)\, Y_{lm}^*(\theta', \phi')$$

(8) Using (7) in (6)

$$\phi(\vec{r}) = \frac{CR^2}{4\pi\epsilon_0} \int_0^{2\pi} d\phi' \int_0^{\pi} \sin\theta' d\theta' \sqrt{\frac{4\pi}{3}}\, Y_{10}(\theta',\phi') \sum_{l=0}^{\infty} \sum_{m=-l}^{l} \frac{r_<^l}{r_>^{l+1}} \frac{4\pi}{2l+1} Y_{lm}(\theta,\phi)\, Y_{lm}^*(\theta',\phi')$$

$$= \frac{CR^2}{4\pi\epsilon_0} \sqrt{\frac{4\pi}{3}} \sum_{l=0}^{\infty} \sum_{m=-l}^{l} \frac{r_<^l}{r_>^{l+1}} \frac{4\pi}{2l+1} Y_{lm}(\theta,\phi) \underbrace{\int_0^{2\pi} d\phi' \int_0^{\pi} \sin\theta' d\theta'\, Y_{10}(\theta',\phi')\, Y_{lm}^*(\theta',\phi')}_{\delta_{l1}\,\delta_{m0}}$$

$$= \frac{CR^2}{4\pi\epsilon_0} \sqrt{\frac{4\pi}{3}} \sum_{l=0}^{\infty} \sum_{m=-l}^{l} \frac{r_<^l}{r_>^{l+1}} \frac{4\pi}{2l+1} Y_{lm}(\theta,\phi)\, \delta_{l1}\, \delta_{m0}$$

$$= \frac{CR^2}{4\pi\epsilon_0} \sqrt{\frac{4\pi}{3}} \frac{r_<}{r_>^2} \frac{4\pi}{3} Y_{10}(\theta,\phi)$$

$$P_1(\cos\theta) = \sqrt{\frac{4\pi}{3}}\, Y_{10}(\theta,\phi)$$

$$= \frac{C\cos\theta}{4\pi\epsilon_0} \frac{4\pi}{3} R^2 \frac{r_<}{r_>^2}$$

$$= \frac{\vec{C}\cdot\hat{r}}{4\pi\epsilon_0} \frac{4\pi}{3} R^2 \frac{r_<}{r_>^2}.$$