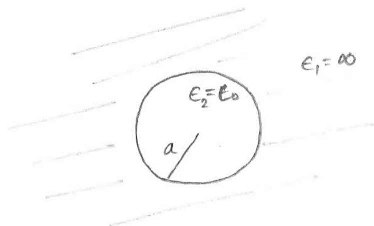
Set - 23

Green's function for a perfectly conducting cylinder (inside)

1 

2 Our goal is to solve the differential equation

$$-\epsilon_0 \nabla^2 G(\vec{r}, \vec{r}') = \delta^{(3)}(\vec{r} - \vec{r}')$$

3
$$G(\vec{r}, \vec{r}') = \frac{1}{\epsilon_0} \int_{-\infty}^{+\infty} \frac{dk_z}{2\pi} e^{ik_z(z-z')} \sum_{m=-\infty}^{+\infty} \frac{1}{2\pi} e^{im(\phi-\phi')} g_m(\rho, \rho'; k_z)$$

4 Substituting 3 in 2 we obtain the differential equation

$$\left[-\frac{1}{\rho}\frac{\partial}{\partial\rho}\rho\frac{\partial}{\partial\rho} + \frac{m^2}{\rho^2} + k_z^2\right] g_m(\rho, \rho'; k_z) = \frac{\delta(\rho-\rho')}{\rho}$$

5 Let us construct

$$g_m(\rho, \rho'; k_z) = \begin{cases} A\, I_m(k_z\rho) + B\, K_m(k_z\rho) & \rho < \rho' \\ C\, I_m(k_z\rho) + D\, K_m(k_z\rho) & \rho' < \rho \end{cases}$$

6 We immediately see that

$$B = 0,$$

because otherwise the solution will blow up at the origin.

7 We have the conditions

(i) $$g_m(\rho, \rho'; k_2) \Big|_{\rho=\rho'-\delta}^{\rho=\rho'+\delta} = 0$$

(ii) $$\rho \frac{\partial}{\partial \rho} g_m(\rho, \rho'; k_2) \Big|_{\rho=\rho'-\delta}^{\rho=\rho'+\delta} = -1$$

(iii) $$g_m(a, \rho'; k_2) = 0 \longrightarrow$$ property of a perfect conductor.

8 Using 7 -(iii) we have.

$$C\, I_m(k_2 a) + D\, K_m(k_2 a) = 0$$

$I_a \equiv I_m(k_2 a)$

$K_a \equiv K_m(k_2 a)$

$$D = -C \frac{I_a}{K_a}.$$

9 Using 8 and 6 in 5

$$g_m(\rho,\rho';k_z) = \begin{cases} A\, I_m(k_z\rho) & \rho < \rho' \\ C\, I_m(k_z\rho) - C\,\frac{I_a}{K_a}\, K_m(k_z\rho) & \rho' < \rho \end{cases}$$

10 Using 9 in 7-(i) and 7-(ii)

$$K_m'(k_z\rho') \times \quad C\, I_m(k_z\rho') - C\,\frac{I_a}{K_a}\, K_m(k_z\rho') - A\, I_m(k_z\rho') = 0 \qquad \text{— (i)}$$

$$K_m(k_z\rho') \times \quad C\, I_m'(k_z\rho') - C\,\frac{I_a}{K_a}\, K_m'(k_z\rho') - A\, I_m'(k_z\rho') = -\frac{1}{k_z\rho'} \qquad \text{— (ii)}$$

11 Subtracting

$$(C - A)\underbrace{\left[I_m(k_z\rho')\, K_m'(k_z\rho') - K_m(k_z\rho')\, I_m(k_z\rho') \right]}_{\text{Wronskian } = -\frac{1}{k_z\rho'}} = \frac{1}{k_z\rho'}\, K_m(k_z\rho')$$

$$A - C = K_m(k_z\rho')$$

12 Using 11 in 10-(i)

$$C\, I_m(k_z\rho') - C\,\frac{I_a}{K_a}\, K_m(k_z\rho') - \left(C + K_m(k_z\rho')\right) I_m(k_z\rho') = 0$$

$$-C\,\frac{I_a}{K_a}\, K_m(k_z\rho') = K_m(k_z\rho')\, I_m(k_z\rho')$$

$$C = -\frac{K_a}{I_a}\, I_m(k_z\rho')$$

13 Using 11 and 12

$$C = -\frac{K_a}{I_a} I_m(k_z \rho')$$

$$A = K_m(k_z \rho') - \frac{K_a}{I_a} I_m(k_z \rho')$$

14 Using 13 in 9

$$g_m(\rho, \rho'; k_z) = \begin{cases} I_m(k_z \rho) K_m(k_z \rho') - \frac{K_a}{I_a} I_m(k_z \rho) I_m(k_z \rho') & \rho < \rho' \\ I_m(k_z \rho') K_m(k_z \rho) - \frac{K_a}{I_a} I_m(k_z \rho) I_m(k_z \rho') & \rho' < \rho \end{cases}$$

$$= I_m(k_z \rho_<) K_m(k_z \rho_>) - \frac{K_a}{I_a} I_m(k_z \rho) I_m(k_z \rho')$$