Hydrogen atom as addition of two angular momentum

① The hydrogen atom is described by

$$H = \frac{p^2}{2\mu} - \frac{Ze^2}{r}$$

$$\frac{d\vec{r}}{dt} = \frac{1}{i\hbar}[\vec{r}, H] = \frac{\partial H}{\partial \vec{p}} = \frac{\vec{p}}{\mu}$$

$$\frac{d\vec{p}}{dt} = \frac{1}{i\hbar}[\vec{p}, H] = -\frac{\partial H}{\partial r} = -\frac{Ze^2\vec{r}}{r^3}$$

② The constants of motion are.

$$\frac{dH}{dt} = 0 \qquad H = \frac{p^2}{2\mu} - \frac{Ze^2}{r}$$

$$\frac{d\vec{L}}{dt} = 0 \qquad \vec{L} = \vec{r}\times\vec{p}$$

$$\frac{d\vec{A}}{dt} = 0 \qquad \vec{A} = \frac{\vec{r}}{r} - \frac{1}{\mu Ze^2}\vec{p}\times\vec{L} + \frac{i\hbar}{\mu Ze^2}\vec{p}$$

③ We also have.

$$A^2 = 1 + \frac{2H(L^2+\hbar^2)}{\mu Z^2e^4}$$

and

$$\vec{A}\cdot\vec{L} = \vec{L}\cdot\vec{A} = 0.$$

4 Let us find the commutation relations between $\vec{L}$ and $\vec{A}$. Since $\vec{A}$ is a vector (because it is constructed out of vectors) we know how it commutes with $\vec{L}$. Thus,

$$\vec{L} \times \vec{L} = i\hbar \vec{L} \qquad [L_i, L_j] = i\hbar \epsilon_{ijk} L_k.$$

$$\vec{A} \times \vec{L} + \vec{L} \times \vec{A} = 2i\hbar \vec{A} \qquad [A_i, L_j] = i\hbar \epsilon_{ijk} A_k$$

$$\vec{A} \times \vec{A} = ? \qquad [A_i, A_j] = ?$$

5 $\vec{A} \times \vec{A}$ is a constant of motion. Further, it is a vector. Thus, it has to constructed out of

$$\vec{A} \times \vec{A} = i\hbar a \vec{L} + i\hbar b \vec{A} + i\hbar c \vec{A} \times \vec{L}.$$

6 The transformation $\vec{r} \to -\vec{r}$ and $\vec{p} \to -\vec{p}$ leaves the Hamiltonian invariant and

$$\vec{A} \to -\vec{A}, \quad \vec{L} \to \vec{L}, \quad \vec{A} \times \vec{L} \to -\vec{A} \times \vec{L}, \quad \vec{A} \times \vec{A} \to \vec{A} \times \vec{A}.$$

Thus, we conclude that $b = 0$ and $c = 0$ in 5.

$$\vec{A} \times \vec{A} = i\hbar a \vec{L}$$

or

$$[A_i, A_j] = i\hbar a \epsilon_{ijk} L_k.$$

7 To determine the constant a we use 3.

$$[\vec{A}, A^2] = \frac{2H}{\mu Z^2 e^4} [\vec{A}, L^2]$$

$$[A_i, A_j A_j] = \frac{2H}{\mu Z^2 e^4} [A_i, L_j L_j]$$

$$A_j [A_i, A_j] + [A_i, A_j] A_j = \frac{2H}{\mu Z^2 e^4} \left(L_j [A_i, L_j] + [A_i, L_j] L_j \right)$$

using 4 and 6

$$A_j \cancel{i\hbar}\, a\, \epsilon_{ijk} L_k + \cancel{i\hbar}\, a\, \epsilon_{ijk} L_k A_j = \frac{2H}{\mu Z^2 e^4} \left(L_j \cancel{i\hbar}\, \epsilon_{ijk} A_k + \cancel{i\hbar}\, \epsilon_{ijk} A_k L_j \right)$$

$$a\, \epsilon_{ijk} (A_j L_k + L_k A_j) = \frac{2H}{\mu Z^2 e^4} \underset{kj}{\epsilon_{ijk}} \left(\underset{j}{A_k} \underset{k}{L_j} + \underset{k}{L_j} \underset{j}{A_k} \right)$$

$$a\, \epsilon_{ijk} (A_j L_k + L_k A_j) = - \frac{2H}{\mu Z^2 e^4} \epsilon_{ijk} (A_j L_k + L_k A_j)$$

$$a = - \frac{2H}{\mu Z^2 e^4} .$$

8 At this stage, let us pretend we do not yet know the eigenvalues of H. Let

$$H = - \frac{\mu Z^2 e^4}{2 \hbar^2} \frac{1}{s^2} \qquad s > 0 .$$

→ s is dimensionless.

→ $H < 0 \Rightarrow$ bound states

$H > 0 \Rightarrow$ scattering states

9 Using 4 to 8 we have the commutation relations

$$\vec{L} \times \vec{L} = i\hbar \vec{L}$$

$$\vec{A} \times \vec{L} + \vec{L} \times \vec{A} = 2i\hbar \vec{A}$$

$$\vec{A} \times \vec{A} = i\hbar \vec{L} \frac{1}{\hbar^2 s^2}$$

$$[L_i, L_j] = i\hbar \epsilon_{ijk} L_k$$

$$[A_i, L_j] = i\hbar \epsilon_{ijk} A_k$$

$$[A_i, A_j] = i\hbar \epsilon_{ijk} L_k \frac{1}{\hbar^2 s^2}$$

10 Let us construct

$$\vec{J}^{(1)} = \frac{1}{2} (\vec{L} + \hbar s \vec{A})$$

$$\vec{J}^{(2)} = \frac{1}{2} (\vec{L} - \hbar s \vec{A})$$

11

$$\vec{J}^{(1)} \times \vec{J}^{(1)} = \frac{1}{4} (\vec{L} + \hbar s \vec{A}) \times (\vec{L} + \hbar s \vec{A})$$

$$= \frac{1}{4} \vec{L} \times \vec{L} + \frac{\hbar^2 s^2}{4} \vec{A} \times \vec{A} + \frac{\hbar s}{4} (\vec{L} \times \vec{A} + \vec{A} \times \vec{L})$$

$$= \frac{1}{4} i\hbar \vec{L} + \frac{1}{4} i\hbar \vec{L} + \frac{\hbar s}{4} 2i\hbar \vec{A}$$

$$= i\hbar \frac{1}{2} (\vec{L} + \hbar s \vec{A})$$

$$= i\hbar \vec{J}^{(1)}$$

(12) $$\vec{J}^{(2)} \times \vec{J}^{(2)} = \frac{1}{4}\left(\vec{L} - \hbar s \vec{A}\right) \times \left(\vec{L} - \hbar s \vec{A}\right)$$

$$= \frac{1}{4}\vec{L}\times\vec{L} + \frac{\hbar^2 s^2}{4}\vec{A}\times\vec{A} - \frac{\hbar s}{4}\left(\vec{L}\times\vec{A} + \vec{A}\times\vec{L}\right)$$

$$= \frac{1}{4} i\hbar\vec{L} + \frac{1}{4} i\hbar\vec{L} - \frac{\hbar s}{4} 2i\hbar\vec{A}$$

$$= i\hbar\,\frac{1}{2}\left(\vec{L} - \hbar s\vec{A}\right)$$

$$= i\hbar\,\vec{J}^{(2)}$$

(13) $$\vec{J}^{(1)} \times \vec{J}^{(2)} + \vec{J}^{(2)} \times \vec{J}^{(1)}$$

$$= \frac{1}{4}\left(\vec{L} + \hbar s\vec{A}\right) \times \left(\vec{L} - \hbar s\vec{A}\right) + \frac{1}{4}\left(\vec{L} - \hbar s\vec{A}\right) \times \left(\vec{L} + \hbar s\vec{A}\right)$$

$$= \frac{1}{4}\vec{L}\times\vec{L} - \frac{\hbar^2 s^2}{4}\vec{A}\times\vec{A} + \frac{\hbar s}{4}\left(\vec{A}\times\vec{L} - \vec{L}\times\vec{A}\right)$$

$$+ \frac{1}{4}\vec{L}\times\vec{L} - \frac{\hbar^2 s^2}{4}\vec{A}\times\vec{A} - \frac{\hbar s}{4}\left(\vec{A}\times\vec{L} - \vec{L}\times\vec{A}\right)$$

$$= \frac{1}{4} i\hbar\vec{L} - \frac{1}{4} i\hbar\vec{L} + \frac{1}{4} i\hbar\vec{L} - \frac{1}{4} i\hbar\vec{L} = 0.$$

(14) $$\vec{J}^{(1)}\cdot\vec{J}^{(1)} = \vec{J}^{(2)}\cdot\vec{J}^{(2)} = \frac{1}{4}\left(\vec{L} \pm \hbar s\vec{A}\right)\cdot\left(\vec{L} \pm \hbar s\vec{A}\right)$$

$$= \frac{L^2}{4} + \frac{\hbar^2 s^2}{4}A^2 \pm \frac{\hbar s}{4}\left(\underbrace{\vec{L}\cdot\vec{A}}_{=0} + \underbrace{\vec{A}\cdot\vec{L}}_{=0}\right)$$

$$= \frac{L^2}{4} + \frac{\hbar^2 s^2}{4}\left(1 - \frac{1}{\hbar^2 s^2}\left(L^2 + \hbar^2\right)\right)$$

$$= \frac{\hbar^2}{4}\left(s^2 - 1\right)$$

15 Using 10 to 14 we have.

$$\vec{J}^{(1)} = \frac{1}{2}\left(\vec{L} + \hbar s \vec{A}\right)$$

$$\vec{J}^{(2)} = \frac{1}{2}\left(\vec{L} - \hbar s \vec{A}\right)$$

$$\vec{J}^{(1)} \times \vec{J}^{(1)} = i\hbar \, \vec{J}^{(1)}$$

$$\vec{J}^{(2)} \times \vec{J}^{(2)} = i\hbar \, \vec{J}^{(2)}$$

$$\vec{J}^{(1)} \times \vec{J}^{(2)} + \vec{J}^{(2)} \times \vec{J}^{(1)} = 0$$

(or)

$$[J_i^{(1)}, J_j^{(1)}] = i\hbar \, \epsilon_{ijk} \, J_k^{(1)}$$

$$[J_i^{(2)}, J_j^{(2)}] = i\hbar \, \epsilon_{ijk} \, J_k^{(2)}$$

$$[J_i^{(1)}, J_j^{(2)}] = 0$$

with

$$\vec{J}^{(1)} \cdot \vec{J}^{(1)} = \vec{J}^{(2)} \cdot \vec{J}^{(2)} = \frac{\hbar^2}{4}(s^2 - 1).$$

Thus, $\vec{J}^{(1)}$ and $\vec{J}^{(2)}$ are two uncoupled angular momentum of same magnitude.

16 We have $j_1 = j_2$ and

$$\hbar^2 j_1(j_1+1) = \hbar^2 j_2(j_2+1) = \frac{\hbar^2}{4}(s^2-1)$$

where $j_1 = j_2 = 0, \frac{1}{2}, 1, \frac{3}{2}, 2, \ldots$

Let $2j_1 + 1 = 2j_2 + 1 = n$. Thus, $n = 1, 2, 3, 4, \ldots$

$$\Rightarrow \quad j_1(j_1+1) = j_2(j_2+1) = \frac{n-1}{2}\,\frac{n+1}{2} = \frac{1}{4}(n^2-1)$$

$$\Rightarrow \quad n = s.$$

17 Thus, we have.

$$\vec{J}^{(1)} = \frac{1}{2}\left(\vec{L} + \hbar n \vec{A}\right)$$

$$\vec{J}^{(2)} = \frac{1}{2}\left(\vec{L} - \hbar n \vec{A}\right)$$

and

$$\vec{L} = \vec{J}^{(1)} + \vec{J}^{(2)} \quad \text{— total angular momentum}$$

$$\hbar n \vec{A} = \vec{J}^{(1)} - \vec{J}^{(2)}$$

18 Thus, total angular momentum states can be used to construct hydrogen atom states. $|n, l, m\rangle$

$$j_1 = j_2 = 0, \tfrac{1}{2}, 1, \tfrac{3}{2}, \dots$$

$$n = 2j_1 + 1 = 2j_2 + 1 = 1, 2, 3, 4, \dots$$

$$|j_1 - j_2| \leq l \leq j_1 + j_2 \Rightarrow 0 \leq l \leq n-1$$

$$-l \leq m \leq l.$$

Multiplicity of each n is $n^2 = (2j_1+1)(2j_2+1)$.

n – energy level
l – angular momentum number
m – magnetic quantum number.

19 The states can be organized as.

$|n, l, m\rangle$

$\underline{n=1}$ multiplicity $= n^2 = 1$

$|1, 0, 0\rangle$

1s

$\underline{n=2}$ multiplicity $= n^2 = 4$

$|2, 1, 1\rangle$
$|2, 1, 0\rangle$ $|2, 0, 0\rangle$
$|2, 1, -1\rangle$

2p 2s

$\underline{n=3}$ multiplicity $= n^2 = 9$

$|3, 2, 2\rangle$
$|3, 2, 1\rangle$ $|3, 1, 1\rangle$
$|3, 2, 0\rangle$ $|3, 1, 0\rangle$ $|3, 0, 0\rangle$
$|3, 2, -1\rangle$ $|3, 1, -1\rangle$
$|3, 2, -2\rangle$

3d 3p 3s

20 Let us consider the ground state, $|1,0,0\rangle$.

For the ground state $n=1$ $(j_1 = j_2 = 0)$. Thus,

$$\vec{J}^{(1)}\,|1,0,0\rangle = 0 \quad \text{and} \quad \vec{J}^{(2)}\,|1,0,0\rangle = 0,$$

which implies, using 17,

$$\vec{L}\,|1,0,0\rangle = 0 \quad \text{and} \quad \vec{A}\,|1,0,0\rangle = 0.$$

21 Let $\psi_{100}(\vec{r}) = \langle \vec{r}\,|\,1,0,0\rangle$.

$$\vec{L}\,|1,0,0\rangle = 0 \quad \Rightarrow \quad \frac{\hbar}{i}\,(\vec{r}\times\vec{\nabla})\,\psi_{100}(\vec{r}) = 0.$$

$$\vec{A}\,|1,0,0\rangle = 0$$

$$\Rightarrow \left[\frac{\vec{r}}{r} - \frac{1}{\mu Z e^2}\,\vec{P}\times\vec{L} + \frac{i\hbar}{\mu Z e^2}\,\vec{P}\right]|1,0,0\rangle = 0$$

$$\Rightarrow \left(\frac{\vec{r}}{r} + \frac{i\hbar}{\mu Z e^2}\,\vec{P}\right)|1,0,0\rangle = 0 \qquad (\text{using } \vec{L}\,|1,0,0\rangle = 0)$$

$$\Rightarrow \left(\frac{\vec{r}}{r} + \frac{\hbar^2}{\mu Z e^2}\,\vec{\nabla}\right)\psi_{100}(\vec{r}) = 0$$

$\frac{\hbar^2}{\mu e^2} = a_0$ = Bohr radius.

(22) To derive the radial dependence we consid.

$$\hat{r} \cdot \left(\frac{\vec{r}}{r} + \frac{a_0}{Z} \vec{\nabla} \right) \psi_{100}(\vec{r}) = 0$$

$$\left(1 + \frac{a_0}{Z} \frac{\partial}{\partial r} \right) \psi_{100}(r) = 0$$

$$\frac{\partial}{\partial r} \psi_{100}(r) = - \frac{Z}{a_0} \psi_{100}(r)$$

$$\psi_{100}(r) = A\, e^{-\frac{Z r}{a_0}}$$

(23) To determine A we use the normalization condition

$$\int d^3r \, |\psi_{100}(r)|^2 = 1.$$

$$\int_0^\infty dx\, x^2 e^{-x} = \Gamma(3) = 2!$$

$$4\pi \int_0^\infty r^2 dr \; A^2 e^{-\frac{2Zr}{a_0}} = 1$$

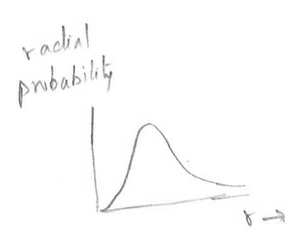
$$4\pi A^2 \left(\frac{a_0}{2Z} \right)^3 \int_0^\infty dx\, x^2 e^{-x} = 1$$

$$4\pi A^2 \left(\frac{a_0}{2Z} \right)^3 2 = 1$$

$$A^2 = \frac{Z^3}{\pi a_0^3}$$

$$A = \left(\frac{Z^3}{\pi a_0^3} \right)^{\frac{1}{2}}.$$

Thus, $$\psi_{100}(r) = \sqrt{\frac{Z^3}{\pi a_0^3}}\; e^{-\frac{Z}{a_0} r}$$



Notice radial probability $= 4\pi r^2 \, |\psi_{100}(r)|^2.$

(24) Consider the Hamiltonian

$$H = \frac{p^2}{2\mu} - \frac{Ze^2}{r}$$

$$H\,|n, l, m\rangle = E_n\,|n, l, m\rangle$$

$$\left[\frac{1}{2\mu}\left(\frac{\hbar}{i}\vec{\nabla}\right)^2 - \frac{Ze^2}{r}\right]\psi_{nlm}(\vec{r}) = E_n\,\psi_{nlm}(\vec{r})$$

$$\left[-\frac{\hbar^2}{2\mu}\nabla^2 - \frac{Ze^2}{r}\right]\psi_{nlm}(\vec{r}) = E_n\,\psi_{nlm}(\vec{r})$$

(25) In spherical polar coordinates we have.

$$\nabla^2 = \frac{1}{r^2}\frac{\partial}{\partial r} r^2 \frac{\partial}{\partial r} + \frac{1}{r^2}\left(\frac{1}{\sin\theta}\frac{\partial}{\partial\theta}\sin\theta\frac{\partial}{\partial\theta} + \frac{1}{\sin^2\theta}\frac{\partial^2}{\partial\phi^2}\right)$$

(26) Remember,

$$L^2\,Y_{lm}(\theta,\phi) = -\hbar^2(\vec{r}\times\vec{\nabla})^2\,Y_{lm} = -\hbar^2\left(\frac{1}{\sin\theta}\frac{\partial}{\partial\theta}\sin\theta\frac{\partial}{\partial\theta} + \frac{1}{\sin^2\theta}\frac{\partial^2}{\partial\phi^2}\right)Y_{lm} = \hbar^2 l(l+1)\,Y_{lm}$$

$$L_z\,Y_{lm}(\theta,\phi) = \frac{\hbar}{i}\frac{\partial}{\partial\phi}Y_{lm}(\theta,\phi) = \hbar m\,Y_{lm}(\theta,\phi)$$

$$L_\pm\,Y_{lm}(\theta,\phi) = \frac{\hbar}{i}\left[\hat{x}\cdot(\vec{r}\times\vec{\nabla}) \pm i\,\hat{y}\cdot(\vec{r}\times\vec{\nabla})\right]Y_{lm}(\theta,\phi)$$

$$= \frac{\hbar}{i}\,e^{\pm i\phi}\left(\pm i\frac{\partial}{\partial\phi} - \cot\theta\frac{\partial}{\partial\phi}\right)Y_{lm}(\theta,\phi)$$

$$= \hbar\sqrt{(l\mp m)(l\pm m+1)}\;Y_{l,m\pm1}(\theta,\phi)$$

(27) Let

$$\psi_{nlm}(r,\theta,\phi) = \sum_{l=0}^{n-1} \sum_{m=-l}^{l} R_{nl}(r)\, Y_{lm}(\theta,\phi)$$

(28)
$$\left[-\frac{\hbar^2}{2\mu}\nabla^2 - \frac{Ze^2}{r}\right]\psi_{nlm}(r,\theta,\phi) = E_n\,\psi_{nlm}(r,\theta,\phi)$$

$$\left[-\frac{\hbar^2}{2\mu}\frac{1}{r^2}\frac{\partial}{\partial r}r^2\frac{\partial}{\partial r} - \frac{\hbar^2}{2\mu}\frac{1}{r^2}\left(\frac{1}{\sin\theta}\frac{\partial}{\partial\theta}\sin\theta\frac{\partial}{\partial\theta} + \frac{1}{\sin^2\theta}\frac{\partial^2}{\partial\phi^2}\right) - \frac{Ze^2}{r}\right]$$

$$\times \sum_{l=0}^{n-1}\sum_{m=-l}^{l} R_{nl}(r)\, Y_{lm}(\theta,\phi) = E_n \sum_{l=0}^{n-1}\sum_{m=-l}^{l} R_{nl}(r)\, Y_{lm}(\theta,\phi)$$

using (26) and completeness of $Y_{lm}(\theta,\phi)$.

$$\left[-\frac{\hbar^2}{2\mu}\frac{1}{r^2}\frac{\partial}{\partial r}r^2\frac{\partial}{\partial r} + \frac{\hbar^2}{2\mu}\frac{1}{r^2}\,l(l+1) - \frac{Ze^2}{r}\right]R_{nl}(r) = E_n\,R_{nl}(r)$$

$$\left[-\frac{1}{r^2}\frac{\partial}{\partial r}r^2\frac{\partial}{\partial r} + \frac{l(l+1)}{r^2} - \frac{2\mu Ze^2}{\hbar^2}\frac{1}{r}\right]R_{nl}(r) = \frac{2\mu}{\hbar^2}E_n\,R_{nl}(r)$$

$$\frac{2\mu E_n}{\hbar^2} = -\frac{\mu^2 Z^2 e^4}{\hbar^4}\frac{1}{n^2} = -\frac{Z^2}{a_0^2}\frac{1}{n^2}$$

(29)
$$\frac{\hbar^2}{\mu e^2} = a_0 \qquad E_n = -\frac{\mu Z^2 e^4}{2\hbar^2}\frac{1}{n^2}$$

$$\left[-\frac{1}{r^2}\frac{\partial}{\partial r}r^2\frac{\partial}{\partial r} + \frac{l(l+1)}{r^2} - \frac{2Z}{a_0 r}\right]R_{nl}(r) = -\frac{Z^2}{a_0^2 n^2}R_{nl}(r)$$

30 Let $x = \frac{Z r}{a_0}$

$$\left[-\frac{1}{x^2}\frac{\partial}{\partial x} x^2 \frac{\partial}{\partial x} + \frac{l(l+1)}{x^2} - \frac{2}{x} \right] R_{nl}(x) = -\frac{1}{n^2} R_{nl}(x)$$

$$\left[\frac{1}{x^2}\frac{\partial}{\partial x} x^2 \frac{\partial}{\partial x} - \frac{l(l+1)}{x^2} + \frac{2}{x} - \frac{1}{n^2} \right] R_{nl}(x) = 0$$

$$\left[\frac{d^2}{dx^2} + \frac{2}{x}\frac{d}{dx} - \frac{l(l+1)}{x^2} + \frac{2}{x} - \frac{1}{n^2} \right] R_{nl}(x) = 0$$

31 For large x the dominant terms are.

$$\left[\frac{d^2}{dx^2} - \frac{1}{n^2} \right] R_{nl}(x) = 0$$

$$\Rightarrow \quad R_{nl}(x) \approx e^{-\frac{x}{n}} \quad \text{for } x \to \infty.$$

32 For small x the dominant terms are.

$$\left[\frac{1}{x^2}\frac{\partial}{\partial x} x^2 \frac{\partial}{\partial x} - \frac{l(l+1)}{x^2} \right] R_{nl}(x) = 0$$

x^l

$x^{l-1+2-1}$

$$\Rightarrow \quad R_{nl}(x) \approx x^l \quad \text{for } x \to 0$$

(33) Thus we look for solutions of the form

$$R_{nl}(x) = x^{l} e^{-\frac{x}{n}} L(x)$$

(34)
$$\frac{d}{dx} R = \frac{d}{dx}\left(x^{l} e^{-\frac{x}{n}} L\right)$$
$$= l x^{l-1} e^{-\frac{x}{n}} L - \frac{1}{n} x^{l} e^{-\frac{x}{n}} L + x^{l} e^{-\frac{x}{n}} \frac{d}{dx} L$$
$$= \frac{l}{x} R - \frac{1}{n} R + x^{l} e^{-\frac{x}{n}} \frac{dL}{dx}$$

(35)
$$\frac{d^2 R}{dx^2} = \frac{d}{dx}\left[\left(\frac{l}{x} - \frac{1}{n}\right) R + x^{l} e^{-\frac{x}{n}} \frac{dL}{dx}\right]$$
$$= -\frac{l}{x^2} R + \left(\frac{l}{x} - \frac{1}{n}\right)\frac{dR}{dx} + l x^{l-1} e^{-\frac{x}{n}} \frac{dL}{dx} - \frac{1}{n} x^{l} e^{-\frac{x}{n}} \frac{dL}{dx} + x^{l} e^{-\frac{x}{n}} \frac{d^2 L}{dx^2}$$
$$= -\frac{l}{x^2} R + \left(\frac{l}{x} - \frac{1}{n}\right)\frac{dR}{dx} + \left(\frac{l}{x} - \frac{1}{n}\right) e^{-\frac{x}{n}} \frac{dL}{dx} + x^{l} e^{-\frac{x}{n}} \frac{d^2 L}{dx^2}$$

(36)
$$\frac{d^2 R}{dx^2} + \frac{2}{x}\frac{dR}{dx} = -\frac{l}{x^2} R + \left(\frac{l}{x} - \frac{1}{n}\right)\frac{dR}{dx} + \left(\frac{l}{x} - \frac{1}{n}\right) x^{l} e^{-\frac{x}{n}} \frac{dL}{dx} + x^{l} e^{-\frac{x}{n}} \frac{d^2 L}{dx^2}$$
$$+ \frac{2l}{x^2} R - \frac{2}{nx} R + \frac{2}{x} x^{l} e^{-\frac{x}{n}} \frac{dL}{dx}$$
$$= \frac{l}{x^2} R - \frac{2}{nx} R + \left(\frac{l+2}{x} - \frac{1}{n}\right) x^{l} e^{-\frac{x}{n}} \frac{dL}{dx} + x^{l} e^{-\frac{x}{n}} \frac{d^2 L}{dx^2}$$
$$+ \left(\frac{l}{x} - \frac{1}{n}\right)\left[\left(\frac{l}{x} - \frac{1}{n}\right) R + x^{l} e^{-\frac{x}{n}} \frac{dL}{dx}\right]$$
$$= x^{l} e^{-\frac{x}{n}}\left[\frac{l}{x^2} - \frac{2}{nx} + \left(\frac{l+2}{x} - \frac{1}{n}\right)\frac{d}{dx} + \frac{d^2}{dx^2} + \left(\frac{l}{x} - \frac{1}{n}\right)\frac{d}{dx} + \left(\frac{l}{x} - \frac{1}{n}\right)^2\right] L$$
$$= x^{l} e^{-\frac{x}{n}}\left[\frac{l(l+1)}{x^2} + \frac{1}{n^2} - \frac{2l}{nx} - \frac{2}{nx} + \left(\frac{2l+2}{x} - \frac{2}{n}\right)\frac{d}{dx} + \frac{d^2}{dx^2}\right] L$$
$$= x^{l} e^{-\frac{x}{n}}\left[\frac{l(l+1)}{x^2} + \frac{1}{n^2} - \frac{2l}{nx} - \frac{2}{nx} + \left(\frac{2l+2}{x} - \frac{2}{n}\right)\frac{d}{dx} + \frac{d^2}{dx^2}\right.$$

37 Using 36 in 30

$$\left[\frac{l(l+1)}{x^2} + \frac{1}{n^2} - \frac{2(l+1)}{nx} + \left(\frac{2(l+1)}{x} - \frac{2}{n}\right)\frac{d}{dx} + \frac{d^2}{dx^2} - \frac{l(l+1)}{x^2} + \frac{2}{x} - \frac{1}{n^2}\right] L = 0$$

$$\frac{d^2L}{dx^2} + \left(\frac{2(l+1)}{x} - \frac{2}{n}\right)\frac{dL}{dx} + \frac{2}{x}\left(1 - \frac{l+1}{n}\right)L = 0$$

$$x\frac{d^2L}{dx^2} + 2\left(l+1 - \frac{x}{n}\right)\frac{dL}{dx} + 2\left(1 - \frac{l+1}{n}\right)L = 0$$

38 $y = \frac{2x}{n}$

$$\frac{2}{n}\, y\frac{d^2L}{dy^2} + 2\left(l+1 - \frac{1}{2}y\right)\frac{2}{n}\frac{dL}{dy} + 2\left(1 - \frac{l+1}{n}\right)L = 0$$

$$y\frac{d^2L}{dy^2} + 2\left(l+1 - \frac{1}{2}y\right)\frac{dL}{dy} + \frac{n}{\cancel{2}}\,\cancel{2}\left(1 - \frac{(l+1)}{n}\right)L = 0$$

$$y\frac{d^2L}{dy^2} + (2l+2-y)\frac{dL}{dy} + (n - (l+1))\,L = 0$$

$$y\frac{d^2L}{dy^2} + \left[(2l+1) + 1 - y\right]\frac{dL}{dy} + \left[(n+l) - (2l+1)\right]L = 0$$

39 Differential equation for Laguerre polynomials is.

$$\left[x \frac{d^2}{dx^2} + (\alpha+1-x)\frac{d}{dx} + n\right] L_n^{(\alpha)}(x) = 0$$

40 Comparing 38 and 39 we have radial wave functn

given in terms of

$$L_{n-l-1}^{2l+1}(y) \qquad y = \frac{2x}{n}, \quad x = \frac{Zr}{a_0}$$

or

$$L_{n-l-1}^{2l+1}\left(\frac{2Zr}{na_0}\right)$$

41 Using 40 in 33

$$R_{nl}(r) = \left(\frac{Zr}{a_0}\right)^l e^{-\frac{Zr}{na_0}} L_{n-l-1}^{2l+1}\left(\frac{2Zr}{na_0}\right)$$

42 Thus, the complete wave functn is.

$$\psi_{nlm}(r,\theta,\phi) = R_{nl}(r)\, Y_{lm}(\theta,\phi).$$