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Addition of angular momentum

① A infinitesimal rotation is generated by the unitary transformation

$$U = 1 + \frac{i}{\hbar}\,\delta\vec{\omega}\cdot\vec{J},$$

where the components of angular momentum $\vec{J}$ satisfies

$$\frac{1}{i\hbar}\,[J_i\,,\,J_j] = \epsilon_{ijk}\,J_k\,.$$

A basis set of angular momentum is constructed using

$$\frac{1}{\hbar^2}\,J^2\,|j,m\rangle = j(j+1)\,|j,m\rangle$$

$J_\pm = J_x + iJ_y$

$$\frac{1}{\hbar}\,J_z\,|j,m\rangle = m\,|j,m\rangle$$

$$\frac{1}{\hbar}\,J_+\,|j,m\rangle = \sqrt{(j-m)(j+m+1)}\;|j,m+1\rangle$$

$$\frac{1}{\hbar}\,J_-\,|j,m\rangle = \sqrt{(j+m)(j-m+1)}\;|j,m-1\rangle$$

2 Consider two parts of a composite system described by a common coordinate system ($\delta\vec{\omega}$ is same for each system). Eg. Two atoms in the same magnetic field.

$$U^{(1)} = 1 + \frac{i}{\hbar}\, \delta\vec{\omega} \cdot \vec{J}^{(1)}$$

$$U^{(2)} = 1 + \frac{i}{\hbar}\, \delta\vec{\omega} \cdot \vec{J}^{(2)}$$

which for infinitesimal rotations implies

$$U^{(1)} U^{(2)} = 1 + \frac{i}{\hbar}\, \delta\vec{\omega} \cdot \left(\vec{J}^{(1)} + \vec{J}^{(2)} \right)$$

3 Thus, the total angular momentum of the composite system is

$$\vec{J} = \vec{J}^{(1)} + \vec{J}^{(2)},$$

where we are considering the scenario in which

$$\left[\vec{J}^{(1)}, \vec{J}^{(2)} \right] = 0.$$

4 In particular

$$\frac{1}{i\hbar} \left[\vec{J}^{(1)}, \delta\vec{\omega} \cdot \left(\vec{J}^{(1)} + \vec{J}^{(2)} \right) \right] = \delta\vec{\omega} \times \vec{J}^{(1)}$$

$$\frac{1}{i\hbar} \left[\vec{J}^{(2)}, \delta\vec{\omega} \cdot \left(\vec{J}^{(1)} + \vec{J}^{(2)} \right) \right] = \delta\vec{\omega} \times \vec{J}^{(2)}$$

uses $\left[\vec{J}^{(1)}, \vec{J}^{(2)} \right] = 0$.

$$\frac{1}{i\hbar} \left[\vec{J}^{(1)} + \vec{J}^{(2)}, \delta\vec{\omega} \cdot \left(\vec{J}^{(1)} + \vec{J}^{(2)} \right) \right] = \delta\vec{\omega} \times \left(\vec{J}^{(1)} + \vec{J}^{(2)} \right).$$

(5) Product of individual state : $|j_1, m_1\rangle\, |j_2, m_2\rangle$

$$| \; \rangle = \sum_{m_1=-j_1}^{j_1} \sum_{m_2=-j_2}^{j_2} A_{m_1, m_2} \, |j_1, m_1\rangle\, |j_2, m_2\rangle$$

Total angular momentum state : $|j, m\rangle$

$$| \; \rangle = \sum_{j=|j_1-j_2|}^{j_1+j_2} \sum_{m=-j}^{j} B_{jm} \, |j, m\rangle$$

Clebsh-Gordan coefficients

$$|j, m\rangle = \sum_{m_1=-j_1}^{j_1} \sum_{m_2=-j_2}^{j_2} C^{j_1, j_2}_{jm;\, m_1 m_2} \, |j_1, m_1\rangle\, |j_2, m_2\rangle$$

(6) Agenda

(i) To identify and organise the product of individual states. They are $(2j_1+1)(2j_2+1)$ in total.

(ii) To identify and organise the total angular momentum states. They are $(2j_1+1)(2j_2+1)$ in total.

(iii) To construct the total angular momentum states.

7 Example: $j_1 = \frac{1}{2}$, $j_2 = \frac{1}{2}$

Product states :

$m_1 + m_2 = 1$ $\quad |\frac{1}{2}, \frac{1}{2}\rangle_{①} |\frac{1}{2}, \frac{1}{2}\rangle_{②}$

$m_1 + m_2 = 0$ $\quad |\frac{1}{2}, -\frac{1}{2}\rangle_{①} |\frac{1}{2}, \frac{1}{2}\rangle_{②}$ $\quad |\frac{1}{2}, \frac{1}{2}\rangle_{①} |\frac{1}{2}, -\frac{1}{2}\rangle_{②}$

$m_1 + m_2 = -1$ $\quad |\frac{1}{2}, -\frac{1}{2}\rangle_{①} |\frac{1}{2}, -\frac{1}{2}\rangle_{②}$.

Totatal angular moment states

$j = 1$ $\quad j = 0$

$|1, 1\rangle$

$|1, 0\rangle$ $\quad |0, 0\rangle$

$|1, -1\rangle$

8 To derive the C G coefficients we shall use

$$J_z = J_z^{(1)} + J_z^{(2)} \Rightarrow m = m_1 + m_2$$

$$J_+ = J_+^{(1)} + J_+^{(2)}$$

$$J_- = J_-^{(1)} + J_-^{(1)}$$

9 For $j_1 = \frac{1}{2}$, $j_2 = \frac{1}{2}$, the largest value of m is $m=1$.

It is unique, because only $m_1 = j_1 = \frac{1}{2}$ and $m_2 = j_2 = \frac{1}{2}$ can produce it. Thus, the largest possible j is $j_1 + j_2$.

$$\underset{\substack{\downarrow \\ j}}{|1}, \underset{\substack{\downarrow \\ m}}{1\rangle} = |\underset{\substack{\downarrow \\ j_1}}{\tfrac{1}{2}}, \underset{\substack{\downarrow \\ m_1}}{\tfrac{1}{2}}\rangle_{1} \; |\underset{\substack{\downarrow \\ j_2}}{\tfrac{1}{2}}, \underset{\substack{\downarrow \\ m_2}}{\tfrac{1}{2}}\rangle_{2}$$

10 $$\frac{1}{\hbar} J_- |1,1\rangle = \left\{ \frac{1}{\hbar} J_-^{(1)} + \frac{1}{\hbar} J_-^{(2)} \right\} |\tfrac{1}{2}, \tfrac{1}{2}\rangle_{1} |\tfrac{1}{2}, \tfrac{1}{2}\rangle_{2}$$

Using $\frac{1}{\hbar} J_- |j, m\rangle = \sqrt{(j+m)(j-m+1)}\; |j, m-1\rangle$

$$\sqrt{2}\; |1, 0\rangle = \frac{1}{\hbar} J_-^{(1)} |\tfrac{1}{2}, \tfrac{1}{2}\rangle_{1} |\tfrac{1}{2}, \tfrac{1}{2}\rangle_{2} + |\tfrac{1}{2}, \tfrac{1}{2}\rangle_{1} \frac{1}{\hbar} J_-^{(2)} |\tfrac{1}{2}, \tfrac{1}{2}\rangle_{2}$$

$$= \sqrt{1}\; |\tfrac{1}{2}, -\tfrac{1}{2}\rangle_{1} |\tfrac{1}{2}, \tfrac{1}{2}\rangle_{2} + \sqrt{1}\; |\tfrac{1}{2}, \tfrac{1}{2}\rangle_{1} |\tfrac{1}{2}, -\tfrac{1}{2}\rangle$$

$$|1, 0\rangle = \frac{1}{\sqrt{2}} |\tfrac{1}{2}, -\tfrac{1}{2}\rangle_{1} |\tfrac{1}{2}, \tfrac{1}{2}\rangle_{2} + \frac{1}{\sqrt{2}} |\tfrac{1}{2}, \tfrac{1}{2}\rangle_{1} |\tfrac{1}{2}, -\tfrac{1}{2}\rangle_{2}$$

11 Applying the lowering operator again we have.

$$\sqrt{2}\; |1, -1\rangle = \frac{1}{\sqrt{2}} |\tfrac{1}{2}, -\tfrac{1}{2}\rangle_{1} |\tfrac{1}{2}, -\tfrac{1}{2}\rangle_{2} + \frac{1}{\sqrt{2}} |\tfrac{1}{2}, -\tfrac{1}{2}\rangle_{1} |\tfrac{1}{2}, -\tfrac{1}{2}\rangle_{2}$$

$$|1, -1\rangle = |\tfrac{1}{2}, -\tfrac{1}{2}\rangle_{1} |\tfrac{1}{2}, -\tfrac{1}{2}\rangle_{2}$$

12 Note that the states $|1,1\rangle$, $|1,0\rangle$, and $|1,-1\rangle$ are normalized. There must be a fourth state. It is

$$|\, j=0, m=0\rangle .$$

It is constructed by observing that there could be an $m=0$ state orthogonal to $|1,0\rangle$.

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$$|1,0\rangle = \frac{1}{\sqrt{2}} \left|\tfrac{1}{2}, -\tfrac{1}{2}\right\rangle_{①} \left|\tfrac{1}{2}, \tfrac{1}{2}\right\rangle_{②} + \frac{1}{\sqrt{2}} \left|\tfrac{1}{2}, \tfrac{1}{2}\right\rangle_{①} \left|\tfrac{1}{2}, -\tfrac{1}{2}\right\rangle_{②}$$

$$|0,0\rangle = a \left|\tfrac{1}{2}, -\tfrac{1}{2}\right\rangle_{①} \left|\tfrac{1}{2}, \tfrac{1}{2}\right\rangle_{②} + b \left|\tfrac{1}{2}, \tfrac{1}{2}\right\rangle_{①} \left|\tfrac{1}{2}, -\tfrac{1}{2}\right\rangle_{②}$$

We require

(i) $\langle 1,0 | 1,0\rangle = 1 \quad \Rightarrow \quad a^2 + b^2 = 1 \quad \Rightarrow \quad a = \frac{1}{\sqrt{2}}, \; b = -\frac{1}{\sqrt{2}}$

(ii) $\langle 1,0 | 0,0\rangle = 0 \quad \Rightarrow \quad \frac{a}{\sqrt{2}} + \frac{b}{\sqrt{2}} = 0$

Thus, we have (upto a phase)

$$|0,0\rangle = \frac{1}{\sqrt{2}} \left|\tfrac{1}{2}, -\tfrac{1}{2}\right\rangle_{①} \left|\tfrac{1}{2}, \tfrac{1}{2}\right\rangle_{②} - \frac{1}{\sqrt{2}} \left|\tfrac{1}{2}, \tfrac{1}{2}\right\rangle_{①} \left|\tfrac{1}{2}, -\tfrac{1}{2}\right\rangle_{②} .$$

14 To describe the pattern leading to the construction of C-G coefficients, let us consider the example $j_1 = 3$, $j_2 = 1$.

Product of individual states (listed by (m_1, m_2) pairs)

			$m_1 + m_2 = 4$
$(3,1)$			3
$(2,1)$	$(3,0)$		2
$(1,1)$	$(2,0)$	$(3,-1)$	1
$(0,1)$	$(1,0)$	$(2,-1)$	0
$(-1,1)$	$(0,0)$	$(1,-1)$	-1
$(-2,1)$	$(-1,0)$	$(0,-1)$	-2
$(-3,1)$	$(-2,0)$	$(-1,-1)$	-3
	$(-3,0)$	$(-2,-1)$	-4
		$(-3,-1)$	

(m_1, m_2)

Total angular momentum states

$\|4,4\rangle$		
$\|4,3\rangle$	$\|3,3\rangle$	
$\|4,2\rangle$	$\|3,2\rangle$	$\|2,2\rangle$
$\|4,1\rangle$	$\|3,1\rangle$	$\|2,1\rangle$
$\|4,0\rangle$	$\|3,0\rangle$	$\|2,0\rangle$
$\|4,-1\rangle$	$\|3,-1\rangle$	$\|2,-1\rangle$
$\|4,-2\rangle$	$\|3,-2\rangle$	$\|2,-2\rangle$
$\|4,-3\rangle$	$\|3,-3\rangle$	
$\|4,-4\rangle$		

15 Total states in product of individual state is

$$\sum_{m_1=-j_1}^{j_1} \sum_{m_2=-j_2}^{j_2} = (2j_1+1)(2j_2+1)$$

Total states in total angular momentum state is (let $j_1 \geq j_2$)

$$\sum_{j=j_1-j_2}^{j_1+j_2} \sum_{m=-j}^{j} = \sum_{j=j_1-j_2}^{j_1+j_2} 2j+1$$

$$= 2\sum_{j=j_1-j_2}^{j_1+j_2} j + \sum_{j=j_1-j_2}^{j_1+j_2}$$

$$= 2\left[\sum_{j=0}^{j_1+j_2} j - \sum_{j=0}^{(j_1-j_2)-1} j\right] + (j_1+j_2) - (j_1-j_2-1)$$

$$= 2\left[\frac{(j_1+j_2)(j_1+j_2+1)}{2} - \frac{(j_1-j_2-1)(j_1-j_2)}{2}\right] + 2j_2+1$$

$$= 4j_1j_2 + j_1+j_2 + j_1-j_2 + 2j_2+1.$$

$$= 4j_1j_2 + 2j_1 + 2j_2 + 1$$

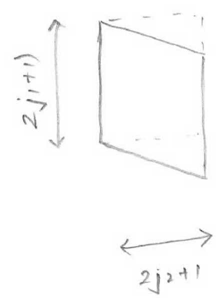
$$= (2j_1+1)(2j_2+1)$$

Note: $\sum_{j=0}^{j_1-j_2-1} j$ term is introduced only if $j_1 > j_2$.

$$\sum_{n=0}^{N} n = \frac{N(N+1)}{2}.$$

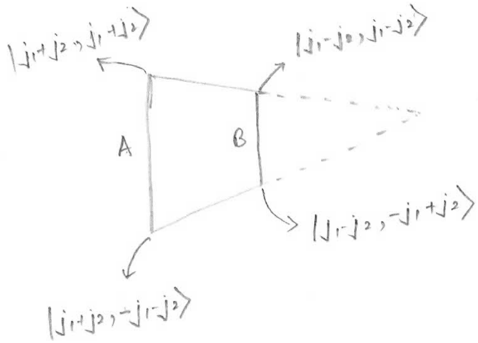
(16) A geometrical proof for product of states is the following (by Alexander Yulaev and Jiayang Xiao, year 2013)

Product of individual states (see (14))



$$\text{Total states} = \text{Area of parrellogram} = (2j_1+1)(2j_2+1)$$

Total angular momentum states (see (14))



Area of trapezium

$$= \frac{1}{2} A h_A - \frac{1}{2} B h_B$$

$$= \frac{1}{2}(A+B)(h_A - h_B) + \underbrace{\frac{1}{2} A h_B - \frac{1}{2} B h_A}_{=0}$$

$$\frac{A}{h_A} = \frac{B}{h_B}$$

$$A = 2(j_1+j_2)+1$$

$$B = 2(j_1-j_2)+1$$

$$h_A - h_B = (j_1+j_2) - (j_1-j_2-1)$$

$$\text{Area} = \frac{1}{2}(4j_1+2)(2j_2+1)$$

$$= (2j_1+1)(2j_2+1).$$

(17) Example: $j_1 = 2$ $j_2 = 1$

Total states $= (2\times 2+1)(2\times 1+1) = 15$.

$\rightarrow$ $|3,3\rangle = |2,2\rangle_{①} |1,1\rangle_{②}$

$\sqrt{6}\, |3,2\rangle = \sqrt{4}\, |2,1\rangle_{①} |1,1\rangle_{②} + \sqrt{2}\, |2,2\rangle_{①} |1,0\rangle_{②}$

$\rightarrow$ $|3,2\rangle = \sqrt{\frac{2}{3}}\, |2,1\rangle_{①} |1,1\rangle_{②} + \sqrt{\frac{1}{3}}\, |2,2\rangle_{①} |1,0\rangle_{②}$

$\sqrt{10}\, |3,1\rangle = \sqrt{\frac{2}{3}}\sqrt{6}\, |2,0\rangle_{①} |1,1\rangle_{②} + \sqrt{\frac{2}{3}}\sqrt{2}\, |2,1\rangle_{①} |1,0\rangle_{②}$

$+ \sqrt{\frac{1}{3}}\sqrt{4}\, |2,1\rangle_{①} |1,0\rangle_{②} + \sqrt{\frac{1}{3}}\sqrt{2}\, |2,2\rangle_{①} |1,-1\rangle_{②}$

$\rightarrow$ $|3,1\rangle = \sqrt{\frac{2}{5}}\, |2,0\rangle_{①} |1,1\rangle_{②} + \sqrt{\frac{8}{15}}\, |2,1\rangle_{①} |1,0\rangle_{②} + \sqrt{\frac{1}{15}}\, |2,2\rangle_{①} |1,-1\rangle_{②}$

Similarly obtain $|3,0\rangle$, $|3,-1\rangle$, $|3,-2\rangle$, and $|3,-3\rangle$.

(18) Let $|2,2\rangle = a\, |2,1\rangle_{①} |1,1\rangle_{②} + b\, |2,2\rangle_{①} |1,0\rangle_{②}$

$a^2 + b^2 = 1$

$a\sqrt{\frac{2}{3}} + b\sqrt{\frac{1}{3}} = 0 \Rightarrow b = -\sqrt{2}\, a$

$\Rightarrow a^2 + 2a^2 = 1 \Rightarrow a = \pm\sqrt{\frac{1}{3}}$ and $b = \mp\sqrt{\frac{2}{3}}$

Thus, upto a phase,

$|2,2\rangle = \sqrt{\frac{1}{3}}\, |2,1\rangle_{①} |1,1\rangle_{②} - \sqrt{\frac{2}{3}}\, |2,2\rangle_{①} |1,0\rangle_{②}$

(19)
$$\sqrt{4}\,|2,1\rangle = \sqrt{\tfrac{1}{3}}\sqrt{6}\,|2,0\rangle_{①}|1,1\rangle_{②} + \sqrt{\tfrac{1}{3}}\sqrt{2}\,|2,1\rangle_{①}|1,0\rangle_{②}$$
$$- \sqrt{\tfrac{2}{3}}\sqrt{4}\,|2,1\rangle_{①}|1,0\rangle_{②} - \sqrt{\tfrac{2}{3}}\sqrt{2}\,|2,2\rangle_{①}|1,-1\rangle_{②}$$

$$\rightarrow \quad |2,1\rangle = \sqrt{\tfrac{1}{2}}\,|2,0\rangle_{①}|1,1\rangle_{②} - \sqrt{\tfrac{1}{6}}\,|2,1\rangle_{①}|1,0\rangle_{②} - \sqrt{\tfrac{1}{3}}\,|2,2\rangle_{①}|1,-1\rangle_{②}$$

Similarly we can construct $|2,0\rangle$, $|2,-1\rangle$, and $|2,-2\rangle$.

(20) Thus,
$$|3,1\rangle = \sqrt{\tfrac{2}{5}}\,|2,0\rangle_{①}|1,1\rangle_{②} + \sqrt{\tfrac{8}{15}}\,|2,1\rangle_{①}|1,0\rangle_{②} + \sqrt{\tfrac{1}{15}}\,|2,2\rangle_{①}|1,-1\rangle_{②}$$
$$|2,1\rangle = \tfrac{1}{\sqrt{2}}\,|2,0\rangle_{①}|1,1\rangle_{②} - \sqrt{\tfrac{1}{6}}\,|2,1\rangle_{①}|1,0\rangle_{②} - \sqrt{\tfrac{1}{3}}\,|2,2\rangle_{①}|1,-1\rangle_{②}$$
$$|1,1\rangle = a\,|2,0\rangle_{①}|1,1\rangle_{②} + b\,|2,1\rangle_{①}|1,0\rangle_{②} + c\,|2,2\rangle_{①}|1,-1\rangle_{②}$$

(21)
$$\langle 1,1|1,1\rangle = 1 \quad \Rightarrow \quad a^2 + b^2 + c^2 = 1.$$
$$\langle 3,1|1,1\rangle = 0 \quad \Rightarrow \quad a\sqrt{6} + b\sqrt{8} + c = 0$$
$$\langle 2,1|1,1\rangle = 0 \quad \Rightarrow \quad a\sqrt{3} - b - c\sqrt{2} = 0$$

(22)
$$a\sqrt{6} + b\sqrt{8} = -c$$
$$a\sqrt{3} - b = c\sqrt{2}.$$

$$-\sqrt{6} - \sqrt{24} = -3\sqrt{6}$$
$$\sqrt{12} + \sqrt{3} = 3\sqrt{3}$$
$$1 - \sqrt{16} = -3$$

$$b = \frac{3\sqrt{3}}{-3\sqrt{6}}\,c = -\frac{1}{\sqrt{2}}\,c.$$

$$a = \frac{-3c}{-3\sqrt{6}} = \frac{1}{\sqrt{6}}\,c.$$

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$$a^2 + b^2 + c^2 = 1$$

$$\frac{c^2}{6} + \frac{c^2}{2} + c^2 = 1$$

$$c^2 = \frac{6}{10} \Rightarrow c = \sqrt{\frac{6}{10}}, \quad b = -\sqrt{\frac{3}{10}}, \quad a = \sqrt{\frac{1}{10}}$$

Thus,

$$|1,1\rangle = \sqrt{\frac{1}{10}}\, |2,0\rangle_{①} |1,1\rangle_{②} - \sqrt{\frac{3}{10}}\, |2,1\rangle_{①} |1,0\rangle_{②} + \sqrt{\frac{6}{10}}\, |2,2\rangle_{①} |1,-1\rangle_{②}.$$

Using lowering operator we can construct $|1,0\rangle$ and $|1,-1\rangle$.