SET - 10

(Rotation) Transformation function

(1) While understanding Stern-Gerlach experiment we calculated the transformation function between rotated basis for $j = \frac{1}{2}$. We would like to determine the corresponding transformation function for general j.

(2) In particular we want to find

$$\langle j,m; r\hat{n}| = \sum_{j',m'} \langle j,m; r\hat{n} | j',m'\rangle \langle j'm'|$$

rotated basis set

Rotation transformation function

original basis set.

$\hat{n}$ — axis of rotation

r — amount of rotation about $\hat{n}$

(3) $\langle j,m; r\hat{n}| = \langle j,m| \, U(r\hat{n})$

$$U(r\hat{n}) = e^{\frac{i}{\hbar} r \hat{n}\cdot\vec{J}}$$

(4) Using earlier notes we have

$$\langle j,m| = \langle 0| \frac{y_+^{j+m}}{\sqrt{(j+m)!}} \frac{y_-^{j-m}}{\sqrt{(j-m)!}}$$

$\frac{1}{\hbar} J_+ = y_+^{\dagger} y_-$

$\frac{1}{\hbar} J_- = y_+ y_-^{\dagger}$

$[y_+, y_+^{\dagger}] = 1$

$[y_-, y_-^{\dagger}] = 1$

(5) We have

$$\langle j, m; r\hat{n}| = \langle j, m| U$$

$$= \langle 0| \frac{y_+^{j+m}}{\sqrt{(j+m)!}} \frac{y_-^{j-m}}{\sqrt{(j-m)!}} U$$

$$= \langle 0| U \frac{(U^{-1} y_+ U)^{j+m}}{\sqrt{(j+m)!}} \frac{(U^{-1} y_- U)^{j-m}}{\sqrt{(j-m)!}}$$

(6) Euler angles specify U in terms of three angles.

<u>Step (i)</u>

$$U_1 = e^{\frac{i}{\hbar} \phi J_z}$$

$$\vec{J}' = U_1^{-1} \vec{J} U_1$$

<u>Step (ii)</u> $U_2 = e^{\frac{i}{\hbar} \theta J_y}$

$$\bar{U}_2 = e^{\frac{i}{\hbar} \theta J_y'}$$

$$= U_1^{-1} e^{\frac{i}{\hbar} \theta J_y} U_1$$

$$= U_1^{-1} U_2 U_1$$

$$\vec{J}'' = \bar{U}_2^{-1} \vec{J}' \bar{U}_2$$

$$= (U_1^{-1} U_2 U_1)^{-1} U_1^{-1} \vec{J} U_1 U_1^{-1} U_2 U_1$$

$$= U_1^{-1} U_2^{-1} U_1 U_1^{-1} \vec{J} U_1 U_1^{-1} U_2 U_1$$

$$= U_1^{-1} U_2^{-1} \vec{J} U_2 U_1$$

$$=$$

(8) How does rotation transform $|0\rangle$?

$$\langle 0| \, U(\psi, \theta, \phi) = \langle 0| \, e^{\frac{i}{\hbar}\psi J_z} \, e^{\frac{i}{\hbar}\theta J_y} \, e^{\frac{i}{\hbar}\phi J_z}$$

$J_x + iJ_y = J_+$

$J_x - iJ_y = J_-$

Using

$$\frac{1}{\hbar} J_z = \frac{1}{2} \left(y_+^\dagger y_+ - y_-^\dagger y_- \right)$$

$$\frac{1}{\hbar} J_y = \frac{1}{2i} \left(y_+^\dagger y_- - y_+ y_-^\dagger \right)$$

we have.

$$\langle 0| \, J_z = \langle 0| \, \frac{\hbar}{2} \left(y_+ y_+^\dagger - y_- y_-^\dagger \right) = 0$$

$$\langle 0| \, J_y = \langle 0| \, \frac{\hbar}{2i} \left(y_+ y_-^\dagger - y_+^\dagger y_- \right) = 0$$

Thus only the first term in the expansion of the exponential contributes to give.

$$\langle 0| \, U(\psi, \theta, \phi) = \langle 0| \, .$$

(9) Using (8) in (5) we have

$$\langle j, m| \, U(\psi, \theta, \phi) = \langle 0| \, \frac{(U^{-1} y_+ U)^{j+m}}{\sqrt{(j+m)!}} \, \frac{(U^{-1} y_- U)^{j-m}}{\sqrt{(j-m)!}}$$

$$= \langle 0| \, \frac{\bar{y}_+^{\,j+m}}{\sqrt{(j+m)!}} \, \frac{\bar{y}_-^{\,j-m}}{\sqrt{(j-m)!}}$$

where $\bar{y}_\pm = U^{-1} y_\pm U$

(10) Let us find $U_{2\times 2}$ connecting $Y_\pm$.

$$\begin{pmatrix} \bar{Y}_+ \\ \bar{Y}_- \end{pmatrix} = U_{2\times 2} \begin{pmatrix} Y_+ \\ Y_- \end{pmatrix}$$

Noting that

$$\langle \tfrac{1}{2}, \tfrac{1}{2} | = \langle 0 | \; Y_+$$

$$\langle \tfrac{1}{2}, -\tfrac{1}{2} | = \langle 0 | \; Y_-$$

we realize that $U_{2\times 2}$ is the transformation function for $j = \frac{1}{2}$, which we found while studying the Stern-Gerlach experiment. Let us evaluate it in terms of Euler angles here.

(11) We have. $\quad Y = \begin{pmatrix} Y_+ \\ Y_- \end{pmatrix}$

$$\bar{Y} = e^{-\frac{i}{\hbar} r \hat{n}\cdot\vec{J}} \; Y \; e^{\frac{i}{\hbar} r \hat{n}\cdot\vec{J}}$$

$$\frac{\partial \bar{Y}}{\partial r} = e^{-\frac{i}{\hbar} r \hat{n}\cdot\vec{J}} \left[-\frac{i}{\hbar} \hat{n}\cdot\vec{J}\, Y + \frac{i}{\hbar} Y \,\hat{n}\cdot\vec{J} \right] e^{\frac{i}{\hbar} r \hat{n}\cdot\vec{J}}$$

$$\frac{1}{i}\frac{\partial \bar{Y}}{\partial r} = e^{-\frac{i}{\hbar} r \hat{n}\cdot\vec{J}} \left[Y, \frac{1}{\hbar} \hat{n}\cdot\vec{J} \right] e^{\frac{i}{\hbar} r \hat{n}\cdot\vec{J}}$$

(12) Using the representation for $\vec{J}$ in terms of $y_\pm$

$$\frac{1}{\hbar}\vec{J} = y^\dagger \tfrac{1}{2}\vec{\sigma}\, y$$

we have

$\sigma = +,-$

$$\left[y_\sigma, \tfrac{1}{\hbar}\hat{n}\cdot\vec{J}\right] = \left[y_\sigma, y^\dagger_{\sigma'} \langle\sigma'|\hat{n}\cdot\tfrac{1}{2}\vec{\sigma}|\sigma''\rangle y_{\sigma''}\right]$$

$$= \underbrace{\left[y_\sigma, y^\dagger_{\sigma'}\right]}_{\delta_{\sigma\sigma'}} \langle\sigma'|\hat{n}\cdot\tfrac{1}{2}\vec{\sigma}|\sigma''\rangle y_{\sigma''}$$

$$+ \; y^\dagger_{\sigma'}\langle\sigma'|\hat{n}\cdot\tfrac{1}{2}\vec{\sigma}|\sigma''\rangle \underbrace{\left[y_\sigma, y_{\sigma''}\right]}_{=0}$$

$$= \delta_{\sigma\sigma'}\langle\sigma'|\hat{n}\cdot\tfrac{1}{2}\vec{\sigma}|\sigma''\rangle y_{\sigma''}$$

$$= \langle\sigma|\hat{n}\cdot\tfrac{1}{2}\vec{\sigma}|\sigma''\rangle y_{\sigma''}$$

which is a statement of

$$\left[y, \tfrac{1}{\hbar}\hat{n}\cdot\vec{J}\right] = \hat{n}\cdot\tfrac{1}{2}\vec{\sigma}\, y$$

(13) Using (12) in (11) we have

$$\frac{1}{i}\frac{\partial\bar{y}}{\partial r} = e^{-\frac{i}{\hbar} r\hat{n}\cdot\vec{J}}\, \hat{n}\cdot\tfrac{1}{2}\vec{\sigma}\, y\, e^{\frac{i}{\hbar} r\hat{n}\cdot\vec{J}}$$

$$= \hat{n}\cdot\tfrac{1}{2}\vec{\sigma}\; e^{-\frac{i}{\hbar} r\hat{n}\cdot\vec{J}}\, y\, e^{\frac{i}{\hbar} r\hat{n}\cdot\vec{J}}$$

$$= \hat{n}\cdot\tfrac{1}{2}\vec{\sigma}\;\bar{y}$$

14 Thus we have

$$\frac{1}{i}\frac{\partial \bar{y}}{\partial r} = \hat{n}\cdot\frac{1}{2}\vec{\sigma}\ \bar{y}$$

$$\frac{d\bar{y}}{\bar{y}} = i\ \hat{n}\cdot\frac{1}{2}\vec{\sigma}\ dr$$

$$\bar{y} = e^{ir\hat{n}\cdot\frac{1}{2}\vec{\sigma}}\ (\text{const})$$

We have $\bar{y} = y$ at $r=0$, which determines

$$\text{const} = y$$

Thus,

$$\bar{y} = e^{ir\hat{n}\cdot\frac{1}{2}\vec{\sigma}}\ y$$

15 Comparing 10 and 14 we have.

$$\begin{pmatrix}\bar{y}_+\\ \bar{y}_-\end{pmatrix} = U_{2\times 2}\begin{pmatrix}y_+\\ y_-\end{pmatrix}$$

$$U_{2\times 2} = e^{ir\hat{n}\cdot\frac{1}{2}\vec{\sigma}},$$

which as pointed earlier should give us the transformation function for $j=\frac{1}{2}$.

(16) Using Euler angles we have

$$U_{2\times2} = \langle \tfrac{1}{2}, m | \, e^{i\psi \frac{1}{2}\sigma_z} \, e^{i\theta \frac{1}{2}\sigma_y} \, e^{i\phi \frac{1}{2}\sigma_z} \, | \tfrac{1}{2}, m' \rangle$$

$$= e^{i\psi m} \langle \tfrac{1}{2}, m | \, e^{i\theta \frac{1}{2}\sigma_y} \, | \tfrac{1}{2}, m' \rangle \, e^{i\phi m'}$$

(17) $$e^{i\theta \frac{1}{2}\sigma_y} = 1 + \left(i\frac{\theta}{2}\sigma_y\right) + \frac{1}{2!}\left(i\frac{\theta}{2}\sigma_y\right)^2 + \frac{1}{3!}\left(i\frac{\theta}{2}\sigma_y\right)^3 + \cdots$$ $\sigma_y^2 = 1$

$$= 1 - \frac{1}{2!}\left(\frac{\theta}{2}\right)^2 + \frac{1}{4!}\left(\frac{\theta}{2}\right)^4 - \cdots$$
$$+ i\sigma_y \left[\frac{\theta}{2} - \frac{1}{3!}\left(\frac{\theta}{2}\right)^3 + \frac{1}{5!}\left(\frac{\theta}{2}\right)^5 - \cdots \right]$$

$$= \cos\frac{\theta}{2} + i\sigma_y \sin\frac{\theta}{2}$$

$$= \begin{pmatrix} \cos\frac{\theta}{2} & \sin\frac{\theta}{2} \\ -\sin\frac{\theta}{2} & \cos\frac{\theta}{2} \end{pmatrix}$$

(18) Using (17) in (16) we have

$$U_{2\times2} = \begin{bmatrix} e^{\frac{i}{2}\psi} \cos\frac{\theta}{2} \, e^{\frac{i}{2}\phi} & e^{\frac{i}{2}\psi} \sin\frac{\theta}{2} \, e^{-\frac{i}{2}\phi} \\ -e^{-\frac{i}{2}\psi} \sin\frac{\theta}{2} \, e^{\frac{i}{2}\phi} & e^{-\frac{i}{2}\psi} \cos\frac{\theta}{2} \, e^{\frac{i}{2}\phi} \end{bmatrix}$$

(19) Using (18) in (15) we have.

$$\bar{y}_+ = e^{\frac{i}{2}\psi}\left[\cos\frac{\theta}{2}\left(e^{\frac{i}{2}\phi}y_+\right) + \sin\frac{\theta}{2}\left(e^{-\frac{i}{2}\phi}y_-\right)\right]$$

$$\bar{y}_- = e^{-\frac{i}{2}\psi}\left[-\sin\frac{\theta}{2}\left(e^{\frac{i}{2}\phi}y_+\right) + \cos\frac{\theta}{2}\left(e^{-\frac{i}{2}\phi}y_-\right)\right]$$

(20) Thus, using (1), (4), (9) and (19) we have.

$$\langle j,m|\, U(\psi,\theta,\phi) = \sum_{j',m'} \langle j,m|\, U(\psi,\theta,\phi)\, |j',m'\rangle \langle j',m'|$$

where

$$\langle j',m'| = \langle 0|\, \frac{y_+^{\,j'+m'}}{\sqrt{(j'+m')!}}\, \frac{y_-^{\,j'-m'}}{\sqrt{(j'-m')!}}$$

$$\langle j,m|\, U(\psi,\theta,\phi) = \langle 0|\, \frac{\bar{y}_+^{\,j+m}}{\sqrt{(j+m)!}}\, \frac{\bar{y}_-^{\,j-m}}{\sqrt{(j-m)!}}$$

$$\bar{y}_+ = e^{\frac{i}{2}\psi}\left[\cos\frac{\theta}{2}\left(e^{\frac{i}{2}\phi}y_+\right) + \sin\frac{\theta}{2}\left(e^{-\frac{i}{2}\phi}y_-\right)\right]$$

$$\bar{y}_- = e^{-\frac{i}{2}\psi}\left[-\sin\frac{\theta}{2}\left(e^{\frac{i}{2}\phi}y_+\right) + \cos\frac{\theta}{2}\left(e^{-\frac{i}{2}\phi}y_-\right)\right]$$

The strategy would be to evaluate the transformation function by matching the like coefficients of y_+ and y_-.

21 Using Euler angles we can write

$$\langle j,m|\, U(\psi,\theta,\phi)\,|j',m'\rangle = \langle j,m|\, e^{\frac{i}{\hbar}\psi J_z}\, e^{\frac{i}{\hbar}\theta J_y}\, e^{\frac{i}{\hbar}\phi J_z}\,|j',m'\rangle$$

$$= e^{im\psi}\,\langle j,m|\, e^{\frac{i}{\hbar}\theta J_y}\,|j',m'\rangle\, e^{im'\phi}$$

22 Also, the transformation function vanishes if $j \neq j'$. This is seen by using $[J^2, J_y] = 0$ to evaluate

$$\langle j,m|\,[J^2, f(J_y)\,|j',m'\rangle = \{j(j+1) - j'(j'+1)\}\,\langle j,m|\, f(J_y)\,|j',m'\rangle$$

$$= 0$$

for arbitrary function, which implies.

$$\langle j,m|\, f(J_y)\,|j',m'\rangle = 0 \quad \text{if} \quad j \neq j'.$$

23 Using 22 in 21 we have.

$$\langle j,m|\, U(\psi,\theta,\phi)\,|j',m'\rangle = \delta_{jj'}\, e^{im\psi}\, \underbrace{\langle j,m|\, e^{\frac{i}{\hbar}\theta J_y}\,|j,m'\rangle}_{U^{(j)}_{m,m'}(\theta)}\, e^{im'\phi}$$

24 Using 23 in 20

$$\langle 0| \, e^{im\psi} \frac{\left\{ \cos\frac{\theta}{2} \left(e^{\frac{i}{2}\phi} y_+\right) + \sin\frac{\theta}{2} \left(e^{-\frac{i}{2}\phi} y_-\right) \right\}^{j+m}}{\sqrt{(j+m)!}} \frac{\left\{ -\sin\frac{\theta}{2} \left(e^{\frac{i}{2}\phi} y_+\right) + \cos\frac{\theta}{2} \left(e^{-\frac{i}{2}\phi} y_-\right) \right\}^{j-m}}{\sqrt{(j-m)!}}$$

$$= \delta_{jj'} \, e^{im\psi} \sum_{m'=-j}^{+j} U^{(j)}_{m,m'}(\theta) \frac{y_+^{j+m'}}{\sqrt{(j+m')!}} \frac{y_-^{j-m'}}{(j-m')!} e^{im'\phi}$$

25 Since the combination $\left(e^{\frac{i}{2}\phi} y_+\right)$ and $\left(e^{-\frac{i}{2}\phi} y_-\right)$ come together on the left it is guaranteed that the factors of $e^{im\phi}$ match on each term. Thus, we can set $\psi = 0$ and $\phi = 0$ for the purpose of evaluating $U^{(j)}_{m,m'}(\theta)$. Thus,

$$\langle 0| \frac{\left\{ \cos\frac{\theta}{2} \, y_+ + \sin\frac{\theta}{2} \, y_- \right\}^{j+m}}{\sqrt{(j+m)!}} \frac{\left\{ -\sin\frac{\theta}{2} \, y_+ + \cos\frac{\theta}{2} \, y_- \right\}^{j-m}}{\sqrt{(j-m)!}}$$

$$= \sum_{m'=-j}^{+j} \langle 0| \, U^{(j)}_{m,m'}(\theta) \frac{y_+^{j+m'}}{\sqrt{(j+m')!}} \frac{y_-^{j-m'}}{\sqrt{(j-m')!}}$$

(26) As an illustrative example let us determine the transformation function for $j = 1$.

(27) Letting $j = 1$, $m = 1$, in (25) we have

$$\frac{1}{\sqrt{2}} \left(\cos \frac{\theta}{2}\, y_+ + \sin \frac{\theta}{2}\, y_- \right)^2 \left(- \sin \frac{\theta}{2}\, y_+ + \cos \frac{\theta}{2}\, y_- \right)^0 = \sum_{m'=-1}^{+1} U^{(1)}_{1,m'}(\theta) \frac{y_+^{1+m'}}{\sqrt{(1+m')!}} \frac{y_-^{1-m'}}{\sqrt{(1-m')!}}$$

$$\frac{1}{\sqrt{2}} \left[\cos^2 \frac{\theta}{2}\, y_+^2 + 2 \sin \frac{\theta}{2} \cos \frac{\theta}{2}\, y_+ y_- + \sin^2 \frac{\theta}{2}\, y_-^2 \right]$$

$$= U^{(1)}_{1,1}(\theta) \frac{1}{\sqrt{2}} y_+^2 + U^{(1)}_{1,0}(\theta)\, y_+ y_- + U^{(1)}_{1,-1}(\theta) \frac{y_-^2}{\sqrt{2}}.$$

which lets us read out

$$U^{(1)}_{1,1}(\theta) = \cos^2 \frac{\theta}{2} \qquad U^{(1)}_{1,0}(\theta) = \sqrt{2} \sin \frac{\theta}{2} \cos \frac{\theta}{2} \qquad U^{(1)}_{1,-1}(\theta) = \sin^2 \frac{\theta}{2}.$$

(28) Letting $j = 1$, $m = 0$, in (25) we have.

$$\left\{ \cos \frac{\theta}{2}\, y_+ + \sin \frac{\theta}{2}\, y_- \right\}^1 \left\{ - \sin \frac{\theta}{2}\, y_+ + \cos \frac{\theta}{2}\, y_- \right\}^1 = \sum_{m'=-1}^{+1} U^{(1)}_{0,m'}(\theta) \frac{y_+^{1+m'}}{\sqrt{(1+m')!}} \frac{y_-^{1-m'}}{\sqrt{(1-m')!}}$$

which leads to

$$U^{(1)}_{0,1}(\theta) = - \sqrt{2} \cos \frac{\theta}{2} \sin \frac{\theta}{2} \qquad U^{(1)}_{0,0}(\theta) = \cos \theta \qquad U^{(1)}_{0,-1}(\theta) = \sqrt{2} \cos \frac{\theta}{2} \sin \frac{\theta}{2}.$$

29 Letting $j=1$, $m=-1$, in 25 we have

$$\frac{1}{\sqrt{2}}\left\{\cos\frac{\theta}{2}\, y_+ + \sin\frac{\theta}{2}\, y_-\right\}^0\left\{-\sin\frac{\theta}{2}\, y_+ + \cos\frac{\theta}{2}\, y_-\right\}^2 = \sum_{m'=-1}^{+1} U^{(j)}_{-1,m'}\frac{y_+^{1+m'}}{\sqrt{(1+m')!}}\,\frac{y_-^{1-m'}}{\sqrt{(1-m')!}}$$

$$\frac{1}{\sqrt{2}}\left[\sin^2\frac{\theta}{2}\, y_+^2 - 2\cos\frac{\theta}{2}\sin\frac{\theta}{2}\, y_+ y_- + \cos^2\frac{\theta}{2}\, y_-^2\right]$$

$$= U^{(1)}_{-1,1}(\theta)\,\frac{1}{\sqrt{2}}\, y_+^2 + U^{(1)}_{-1,0}(\theta)\, y_+ y_- + U^{(1)}_{-1,-1}(\theta)\,\frac{1}{\sqrt{2}}\, y_-^2$$

which gives us

$$U^{(1)}_{-1,1}(\theta) = \sin^2\frac{\theta}{2} \qquad U^{(1)}_{-1,0}(\theta) = -\sqrt{2}\cos\frac{\theta}{2}\sin\frac{\theta}{2} \qquad U^{(1)}_{-1,-1}(\theta) = \cos^2\frac{\theta}{2}$$

30 Using 27, 28, and 29 in 23 we have the transformation function for $j=1$ as

$$\langle j,m|\, U(\psi,\theta,\phi)\,|j,m'\rangle = \begin{bmatrix} e^{i\psi}\cos^2\frac{\theta}{2}\, e^{i\phi} & e^{i\psi}\sqrt{2}\cos\frac{\theta}{2}\sin\frac{\theta}{2} & e^{i\psi}\sin^2\frac{\theta}{2}\, e^{-i\phi} \\ -\sqrt{2}\cos\frac{\theta}{2}\sin\frac{\theta}{2}\, e^{i\phi} & \cos\theta & \sqrt{2}\cos\frac{\theta}{2}\sin\frac{\theta}{2}\, e^{-i\phi} \\ e^{-i\psi}\sin^2\frac{\theta}{2}\, e^{i\phi} & -e^{-i\psi}\sqrt{2}\cos\frac{\theta}{2}\sin\frac{\theta}{2} & e^{-i\psi}\cos^2\frac{\theta}{2}\, e^{-i\phi} \end{bmatrix}$$

(31) Thus, the probabilities relating measurements of angular momentum, or magnetic moment, in two different directions related by angle θ is

$$p(m, m'; \theta) = |\langle j, m | U(\psi, \theta, \phi) | j, m' \rangle|^2 = |U^{(j)}_{m,m'}(\theta)|^2$$

$$= \begin{bmatrix} \cos^4 \frac{\theta}{2} & \frac{1}{2} \sin^2 \theta & \sin^4 \frac{\theta}{2} \\ \frac{1}{2} \sin^2 \theta & \cos^2 \theta & \frac{1}{2} \sin^2 \theta \\ \sin^4 \frac{\theta}{2} & \frac{1}{2} \sin^2 \theta & \cos^4 \frac{\theta}{2} \end{bmatrix}$$

(32) Observe that

(i) $p(m, m') = p(m', m)$

(ii) $\sum_m p(m, m') = 1$

(iii) $\sum_{m'} p(m, m') = 1$