Galilean invariance (contd.)

I Variation in an operator due to an infinitesimal unitary transformation (generated by G)

① $X\,|\rangle_a = |\rangle_b$

$U^{-1}|\rangle_a = |\rangle'_a$

$U^{-1}|\rangle_b = |\rangle'_b$

$X'\,|\rangle'_a = |\rangle'_b$

$X' = X - \delta X$

$\Rightarrow \quad X' = U^{-1} X U$

② $$X - \delta X = \left(1 - \frac{i}{\hbar} G\right) X \left(1 + \frac{i}{\hbar} G\right) + O(G)^2.$$

$$= X - \frac{1}{i\hbar}[X, G]$$

$$\Rightarrow \quad \delta X = \frac{1}{i\hbar}[X, G]$$

③ For example, using ㉓ in Set-04

$$\underbrace{\delta_\epsilon \vec{J}}_{\substack{\text{variation of the}\\ \text{operator } \vec{J} \text{ due}\\ \text{to infinitesimal translation}\\ \text{of origin.}}} = \frac{1}{i\hbar}\left[\vec{J}, \vec{P}\cdot\delta\vec{\epsilon}\right] = \delta\vec{\epsilon} \times \vec{P}$$

$\delta\vec{\epsilon}$: infinitesimal translation in origin.

$\vec{P}$: generator of translation in origin.

II Angular momentum: $\vec{J} = \vec{R} \times \vec{P} + \vec{S}$

4 Let us introduce the position operator $\vec{R}$. If we require the Heisenberg's commutation relations

$$\frac{1}{i\hbar}\left[\vec{R}_i, \vec{P}_j\right] = \delta_{ij}, \quad \text{and} \quad \frac{1}{i\hbar}\left[\vec{R}_i, \vec{R}_j\right] = 0$$

we can interpret it as (using 2) the statement of transformation of the operator $\vec{R}$ under infinitesimal translation

$$\delta_e \vec{R} = \frac{1}{i\hbar}\left[\vec{R}, \delta\vec{e}\cdot\vec{P}\right] = \delta\vec{e}.$$

Thus, under infinitesimal translations we have

$$\vec{R}' = \vec{R} - \delta\vec{e}.$$

5 Let us next consider the orbital angular momentum operator

$$\vec{L} = \vec{R} \times \vec{P}.$$

Under translations we have

$$\begin{aligned}
\delta_e \vec{L} &= \frac{1}{i\hbar}\left[\vec{L}, \delta\vec{e}\cdot\vec{P}\right] \\
&= \frac{1}{i\hbar}\vec{R} \times \underbrace{\left[\vec{P}, \delta\vec{e}\cdot\vec{P}\right]}_{=0} + \frac{1}{i\hbar}\left[\vec{R}, \delta\vec{e}\cdot\vec{P}\right] \times \vec{P} \\
&= \frac{1}{i\hbar}\,\delta\vec{e} \times \vec{P}.
\end{aligned}$$

Thus, the orbital angular momentum $\vec{L}$ does transform like angular momentum $\vec{J}$ under translations.

⑥ Let us check the transformation of $\vec{L}$ under rotations, (assuming $\vec{J} = \vec{L}$ temporarily)

$$\delta_\omega \vec{L} = \frac{1}{i\hbar}\left[\vec{L}, \ \delta\vec{\omega}\cdot\vec{L}\right]$$

$$= \frac{1}{i\hbar}\left[\vec{R}\times\vec{P}, \ \delta\vec{\omega}\cdot\vec{R}\times\vec{P}\right]$$

$$= \frac{1}{i\hbar}\left[\epsilon_{ijk} R_j P_k, \ \epsilon_{lmn}\,\delta\omega_l\, R_m P_n\right]$$

$$= \frac{1}{i\hbar}\left[\epsilon_{ijk} R_j, \ \epsilon_{lmn}\,\delta\omega_l\, R_m P_n\right] P_k + \frac{1}{i\hbar} R_j\left[\epsilon_{ijk} P_k, \epsilon_{lmn}\,\delta\omega_l\, R_m P_n\right]$$

$$= \frac{1}{i\hbar}\left[\epsilon_{ijk} R_j, \ \epsilon_{lmn}\,\delta\omega_l\, R_m P_n\right] P_k + \frac{1}{i\hbar} R_j\left[\epsilon_{ijk} P_k, \epsilon_{lmn}\,\delta\omega_l\, R_m\right] P_n$$

used $[R_i, R_j]=0$, $[P_i, P_j]=0 \rightarrow$
$$= \frac{1}{i\hbar} R_m\left[\epsilon_{ijk} R_j, \ \epsilon_{lmn}\,\delta\omega_l\, P_n\right] P_k + \frac{1}{i\hbar} R_j\left[\epsilon_{ijk} P_k, \epsilon_{lmn}\,\delta\omega_l\, R_m\right] P_n$$

used $[R_i, P_j] = i\hbar\,\delta_{ij} \rightarrow$
$$= \delta\omega_l\, R_m\, \epsilon_{ijk}\,\epsilon_{lmn}\,\delta_{jn} P_k - R_j\,\epsilon_{ijk}\,\epsilon_{lmn}\,\delta\omega_l\,\delta_{km} P_n.$$

$$= \epsilon_{jki}\,\epsilon_{jlm}\,\delta\omega_l\, R_m P_k - \epsilon_{kij}\,\epsilon_{knl}\,\delta\omega_l\, R_j P_n.$$

$$= \delta\omega_k\, R_i P_k - \delta\omega_i\, R_k P_k - \delta\omega_j\, R_j P_i + \delta\omega_i\, R_j P_j$$

$$= \delta\vec{\omega}\times(\vec{R}\times\vec{P}).$$

Thus, orbital angular momentum $\vec{L}$ does transform like angular momentum $\vec{J}$ under rotations.

7 What is wrong (or limitations of)

$$\vec{L} = \vec{R} \times \vec{P}$$

being interpreted as the angular momentum?

8 Note that, we have

$$\vec{R} \cdot \vec{L} = \vec{R} \cdot (\vec{R} \times \vec{P}) = 0,$$

always. This implies that 0 is always an eigenvalue of $\vec{L}$. Because for a rotation about the axis $\delta\vec{\omega} \propto \vec{R}$ we will have

$$\langle \,|\, \vec{L} \cdot \delta\vec{\omega} = 0.$$

9 The eigen value spectrum are. $|j\, m\rangle$

$j \geq 0$

$m: \quad -j, -j+1, -j+2, \ldots, +j$

eg: $j = 0: \quad m = 0$

$j = \frac{1}{2}: \quad m = -\frac{1}{2}, \frac{1}{2}$

$j = 1: \quad m = -1, 0, 1$

$j = \frac{3}{2}: \quad m = -\frac{3}{2}, -\frac{1}{2}, \frac{1}{2}, \frac{3}{2}$

10 Thus, half integral spins are not allowed by $\vec{L} = \vec{R} \times \vec{P}$.

III Boost operator : $\vec{N} = \vec{P}t - M\vec{R}$

(13) Under translations in time and space we have.

$$\delta_t \vec{N} = \frac{1}{i\hbar}\left[\vec{N}, -H\delta t\right] = \vec{P}\,\delta t \qquad t' = t - \delta t$$

and

$$\delta_e \vec{N} = \frac{1}{i\hbar}\left[\vec{N}, \delta\vec{e}\cdot\vec{P}\right] = -M\,\delta\vec{e} \qquad \begin{array}{l} \vec{x}' = \vec{x} - \delta\vec{e} \\ \vec{R}' = \vec{R} - \underbrace{\delta\vec{R}}_{=\delta\vec{e}} \end{array}$$

(14) The above equations can be symbolically read as.

$$\frac{\partial \vec{N}}{\partial t} = \frac{1}{i\hbar}\left[\vec{N}, -H\right] = \vec{P}$$

and

$$\frac{\partial \vec{N}}{\partial \vec{R}} = \frac{1}{i\hbar}\left[\vec{N}, \vec{P}\right] = -M\,\overleftrightarrow{1}$$

(15) Interpreting these as operator differential equations, we can deduce.

$$\vec{N} = \vec{P}t - M\vec{R}$$

(16) It can be verified that this construct satisfies the other two commutations relations

$$\delta_\omega \vec{N} = \frac{1}{i\hbar}\left[\vec{N}, \delta\vec{\omega}\cdot\vec{J}\right] = \delta\vec{\omega}\times\vec{N}$$

and

$$\delta_v \vec{N} = \frac{1}{i\hbar}\left[\vec{N}, \delta\vec{v}\cdot\vec{N}\right] = 0.$$

17 Thus, $\vec{N}$ is specified in terms of $\vec{R}$ and $\vec{P}$. This could be the reason why we do not see $\vec{N}$ in non-relativistic quantum mechanics.