SET-04

Galilean invariance

1. Consider the transformation

$$t' = t - \underbrace{\delta t}_{\text{translation in time}}$$

$$\vec{x}' = \vec{x} - \underbrace{\delta\vec{\epsilon}}_{\text{translation in space}} - \underbrace{\delta\vec{\omega}\times\vec{x}}_{\text{rotation in space}} - \underbrace{t\,\delta\vec{v}}_{\text{boost}}$$

2. The unitary operator associated with the above transformation is written in terms of generators:

$$U = 1 + \frac{i}{\hbar} G$$

$$G = -H\,\delta t + \vec{P}\cdot\delta\vec{\epsilon} + \vec{J}\cdot\delta\vec{\omega} + \vec{N}\cdot\delta\vec{v} + \hbar\,\delta\phi$$

where

H : energy ; Hamitonian

$\vec{P}$: linear momentum

$\vec{J}$: angular momentum

$\vec{N}$: generator of boost

and $\delta\phi$ takes care of the fact that an unitary operator is unspecified upto a phase,

$U \to U e^{i\delta\phi}$.

(3) To find the commutation relations, and the transformation properties as we will see, of the generators we consider two successive infinitesimal transformations:

$$t_1 = t_0 - \delta_1 t$$

$$t_{21} = t_1 - \delta_2 t$$
$$= (t_0 - \delta_1 t) - \delta_2 t$$

$$\vec{x}_1 = \vec{x}_0 - \delta_1 \vec{\epsilon} - \delta_1 \vec{\omega} \times \vec{x}_0 - t_0 \delta_1 \vec{v}$$

$$\vec{x}_{21} = \vec{x}_1 - \delta_2 \vec{\epsilon} - \delta_2 \vec{\omega} \times \vec{x}_1 - t_1 \delta_2 \vec{v}$$
$$= (\vec{x}_0 - \delta_1 \vec{\epsilon} - \delta_1 \vec{\omega} \times \vec{x}_0 - t_0 \delta_1 \vec{v}) - \delta_2 \vec{\epsilon} - (t_0 - \delta_1 t)\, \delta_2 \vec{v}$$
$$- \delta_2 \vec{\omega} \times (\vec{x}_0 - \delta_1 \vec{\epsilon} - \delta_1 \vec{\omega} \times \vec{x}_0 - t_0 \delta_1 \vec{v})$$
$$= \vec{x}_0 - \delta_1 \vec{\epsilon} - \delta_2 \vec{\epsilon} - \delta_1 \vec{\omega} \times \vec{x}_0 - \delta_2 \vec{\omega} \times \vec{x}_0 - t_0 \delta_1 \vec{v} - t_0 \delta_2 \vec{v}$$
$$+ \delta_2 \vec{\omega} \times \delta_1 \vec{\epsilon} + \delta_2 \vec{\omega} \times (\delta_1 \vec{\omega} \times \vec{x}_0) + t_0 \delta_2 \vec{\omega} \times \delta_1 \vec{v} + \delta_1 t\, \delta_2 \vec{v}$$

(4) Similarly $(1 \leftrightarrow 2)$

$$t_{12} = t_0 - \delta_2 t - \delta_1 t$$

$$\vec{x}_{12} = \vec{x}_0 - \delta_2 \vec{\epsilon} - \delta_1 \vec{\epsilon} - \delta_2 \vec{\omega} \times \vec{x}_0 - \delta_1 \vec{\omega} \times \vec{x}_0 - t_0 \delta_2 \vec{v} - t_0 \delta_1 \vec{v}$$
$$+ \delta_1 \vec{\omega} \times \delta_2 \vec{\epsilon} + \delta_1 \vec{\omega} \times (\delta_2 \vec{\omega} \times \vec{x}_0) + t_0 \delta_1 \vec{\omega} \times \delta_2 \vec{v} + \delta_2 t\, \delta_1 \vec{v}$$

(5) Subtracting (3) & (4)

$$t_{21} = t_{12}$$

$$\vec{x}_{21} = \vec{x}_{12} - [\delta_1 \vec{\omega} \times (\delta_2 \vec{\omega} \times \vec{x}_0) - \delta_2 \vec{\omega} \times (\delta_1 \vec{\omega} \times \vec{x}_0)]$$
$$- t_0 [\delta_1 \vec{\omega} \times \delta_2 \vec{v} - \delta_2 \vec{\omega} \times \delta_1 \vec{v}]$$
$$- [(\delta_1 \vec{\omega} \times \delta_2 \vec{\epsilon} - \delta_2 \vec{\omega} \times \delta_1 \vec{\epsilon}) + (\delta_2 t\, \delta_1 \vec{v} - \delta_1 t\, \delta_2 \vec{v})]$$

6 Comparing 5 with

$$t_{21} = t_{12} - \delta_{[1,2]} t$$

$$\vec{x}_{21} = \vec{x}_{12} - \delta_{[1,2]} \vec{\epsilon} - \delta_{[1,2]} \vec{\omega} \times \vec{x}_{12} - t_{12}\, \delta_{[1,2]} \vec{V}$$

we identify

$$\delta_{[1,2]} t = 0$$

$$\delta_{[1,2]} \vec{\epsilon} = \delta_1 \omega \times \delta_2 \vec{\epsilon} - \delta_2 \vec{\omega} \times \delta_1 \vec{\epsilon} + \delta_2 t\, \delta_1 \vec{v} - \delta_1 t\, \delta_2 \vec{V}$$

$$\delta_{[1,2]} \vec{\omega} = \delta_1 \vec{\omega} \times \delta_2 \vec{\omega}$$

$$\delta_{[1,2]} \vec{V} = \delta_1 \vec{\omega} \times \delta_2 \vec{V} - \delta_2 \vec{\omega} \times \delta_1 \vec{V}$$

7 To obtain the commutation we will we

$$\frac{1}{i\hbar} [G_1 , G_2] = G_{[1,2]} ,$$

which using 6 reads

$$\frac{1}{i\hbar} \Big[-H \delta_1 t + \vec{P} . \delta_1 \vec{\epsilon} + \vec{J} . \delta_1 \vec{\omega} + \vec{N} \cdot \delta_1 \vec{V} + \hbar \delta_1 \phi ,$$

$$-H \delta_2 t + \vec{P} . \delta_2 \vec{\epsilon} + \vec{J} . \delta_2 \vec{\omega} + \vec{N} . \delta_2 \vec{V} + \hbar \delta_2 \phi \Big]$$

$$= -H \delta_{[1,2]} t + \vec{P} . \delta_{[1,2]} \vec{\epsilon} + \vec{N} . \delta_{[1,2]} \vec{V} + \hbar\, \delta_{[1,2]} \phi .$$

8 What about $\delta_{[1,2]}\phi$? We must have $\delta_{[1,2]}\phi$ to be antisymmetric, $\delta_{[1,2]}\phi = -\delta_{[2,1]}\phi$, because the the commutator on the left hand side is antisymmetric under $1 \leftrightarrow 2$. Further $\delta_{[1,2]}\phi$ must be a scalar because G is constructed to be a scalar.

Most general bilinear combination would have.

$$\delta_1\vec{\epsilon}\cdot\delta_2\vec{\epsilon}\ ,\quad \delta_1\vec{\epsilon}\cdot\delta_2\vec{\omega}\ ,\quad \delta_1\vec{\epsilon}\cdot\delta_2\vec{v}\ ,$$
$$\delta_1\vec{\omega}\cdot\delta_2\vec{\epsilon}\ ,\quad \delta_1\vec{\omega}\cdot\delta_2\vec{\omega}\ ,\quad \delta_1\vec{\omega}\cdot\delta_2\vec{v}\ ,$$
$$\delta_1\vec{v}\cdot\delta_2\vec{\epsilon}\ ,\quad \delta_1\vec{v}\cdot\delta_2\vec{\omega}\ ,\quad \delta_1\vec{v}\cdot\delta_2\vec{v}$$

But the diagonal terms in the above structure cannot construct an antisymmetric structure. Thus,

$$\begin{aligned}\hbar\,\delta_{[1,2]}\phi = \ & K\,(\delta_1\vec{\omega}\cdot\delta_2\vec{\epsilon} - \delta_2\vec{\omega}\cdot\delta_1\vec{\epsilon})\\ & + L\,(\delta_1\vec{\omega}\cdot\delta_2\vec{v} - \delta_2\vec{\omega}\cdot\delta_1\vec{v})\\ & + M\,(\delta_1\vec{\epsilon}\cdot\delta_2\vec{v} - \delta_2\vec{\epsilon}\cdot\delta_1\vec{v})\end{aligned}$$

where K, L, and M are constants.

13 Since 9 is true for arbitrary infinitesimal transformations, 9, 10, 11 & 12 can be satisfied only if $K=0$ and $L=0$. Thus we have the form for $\delta_{[1,2]}\phi$ in 8 to be

$$\hbar\, \delta_{[1,2]}\phi = M\,(\delta_1\vec{\epsilon}\cdot\delta_2\vec{v} - \delta_2\vec{\epsilon}\cdot\delta_1\vec{v})$$

Note that M has dimensions of mass.

15 Using 13 in 7 we have

$$\frac{1}{i\hbar}\left[-H\delta_1 t + \vec{P}\cdot\delta_1\vec{\epsilon} + \vec{J}\cdot\delta_1\vec{\omega} + \vec{N}\cdot\delta_1\vec{v},\ -H\delta_2 t + \vec{P}\cdot\delta_2\vec{\epsilon} + \vec{J}\cdot\delta_2\vec{\omega} + \vec{N}\cdot\delta_2\vec{v}\right]$$

$$= -H\,\delta_{[1,2]}t + \vec{P}\cdot\delta_{[1,2]}\vec{\epsilon} + \vec{N}\cdot\delta_{[1,2]}\vec{v} + M\,(\delta_1\vec{\epsilon}\cdot\delta_2\vec{v} - \delta_2\vec{\epsilon}\cdot\delta_1\vec{v})$$

16 Using 6 in 15

$$\frac{1}{i\hbar}\left[-H\delta_1 t + \vec{P}\cdot\delta_1\vec{\epsilon} + \vec{J}\cdot\delta_1\vec{\omega} + \vec{N}\cdot\delta_1\vec{v},\ -H\delta_2 t + \vec{P}\cdot\delta_2\vec{\epsilon} + \vec{J}\cdot\delta_2\vec{\omega} + \vec{N}\cdot\delta_2\vec{v}\right]$$

$$\begin{aligned}= & -H\cdot 0 \\ & + \vec{P}\cdot(\delta_1\vec{\omega}\times\delta_2\vec{\epsilon} - \delta_2\vec{\omega}\times\delta_1\vec{\epsilon} + \delta_2 t\,\delta_1\vec{v} - \delta_1 t\,\delta_2\vec{v}) \\ & + \vec{J}\cdot\delta_1\vec{\omega}\times\delta_2\vec{\omega} \\ & + \vec{N}\cdot(\delta_1\vec{\omega}\times\delta_2\vec{v} - \delta_2\vec{\omega}\times\delta_1\vec{v}) \\ & + M\,(\delta_1\vec{\epsilon}\cdot\delta_2\vec{v} - \delta_2\vec{\epsilon}\cdot\delta_1\vec{v})\end{aligned}$$

17 Since 16 is true for arbitrary infinitesimal transformations we can compare there coefficients & read out the commutation relations

18 With Hamiltonian:

$$\frac{1}{i\hbar}\left[H\,\delta_1 t\,,\ H\,\delta_2 t \right] = 0$$

$$\frac{1}{i\hbar}\left[\vec{P}\cdot\delta_1\vec{\epsilon}\,,\ H\,\delta_2 t \right] = 0$$

$$\frac{1}{i\hbar}\left[\vec{J}\cdot\delta_1\vec{\omega}\,,\ H\,\delta_2 t \right] = 0$$

$$\frac{1}{i\hbar}\left[\vec{N}\cdot\delta_1\vec{v}\,,\ H\,\delta_2 t \right] = -\vec{P}\cdot\delta_1\vec{v}\;\delta_2 t$$

19 With momentum (under translation)

$$\frac{1}{i\hbar}\left[H\,\delta_1 t\,,\ \vec{P}\cdot\delta_2\vec{\epsilon} \right] = 0$$

$$\frac{1}{i\hbar}\left[\vec{P}\cdot\delta_1\vec{\epsilon}\,,\ \vec{P}\cdot\delta_2\vec{\epsilon} \right] = 0$$

$$\frac{1}{i\hbar}\left[\vec{J}\cdot\delta_1\vec{\omega}\,,\ \vec{P}\cdot\delta_2\vec{\epsilon} \right] = \vec{P}\cdot\delta_1\vec{\omega}\times\delta_2\vec{\epsilon}$$

$$\frac{1}{i\hbar}\left[\vec{N}\cdot\delta_1\vec{v}\,,\ \vec{P}\cdot\delta_2\vec{\epsilon} \right] = -M\,\delta_1\vec{v}\cdot\delta_2\vec{\epsilon}$$

20 With angular momentum (under rotations)

$$\frac{1}{i\hbar}\left[H\,\delta_1 t \,,\ \vec{J}\cdot\delta_2\vec{\omega}\right] = 0$$

$$\frac{1}{i\hbar}\left[\vec{P}\cdot\delta_1\vec{\epsilon} \,,\ \vec{J}\cdot\delta_2\vec{\omega}\right] = -\vec{P}\cdot(\delta_2\vec{\omega}\times\delta_1\vec{\epsilon})$$

$$\frac{1}{i\hbar}\left[\vec{J}\cdot\delta_1\vec{\omega} \,,\ \vec{J}\cdot\delta_2\vec{\omega}\right] = \vec{J}\cdot\delta_1\vec{\omega}\times\delta_2\vec{\omega}$$

$$\frac{1}{i\hbar}\left[\vec{N}\cdot\delta_1\vec{v} \,,\ \vec{J}\cdot\delta_2\vec{\omega}\right] = -\vec{N}\cdot\delta_2\vec{\omega}\times\delta_1\vec{v}$$

21 With generator for boost:

$$\frac{1}{i\hbar}\left[H\,\delta_1 t \,,\ \vec{N}\cdot\delta_2\vec{v}\right] = \vec{P}\cdot\delta_2\vec{v}\ \delta_1 t$$

$$\frac{1}{i\hbar}\left[\vec{P}\cdot\delta_1\vec{\epsilon} \,,\ \vec{N}\cdot\delta_2\vec{v}\right] = M\,\delta_1\vec{\epsilon}\cdot\delta_2\vec{v}$$

$$\frac{1}{i\hbar}\left[\vec{J}\cdot\delta_1\vec{\omega} \,,\ \vec{N}\cdot\delta_2\vec{v}\right] = \vec{N}\cdot\delta_1\vec{\omega}\times\delta_2\vec{v}$$

$$\frac{1}{i\hbar}\left[\vec{N}\cdot\delta_1\vec{v} \,,\ \vec{N}\cdot\delta_2\vec{v}\right] = 0$$

(22) Translation in time:

$$\frac{1}{i\hbar}\left[H, H\,\delta t\right] = 0 = \delta_t H$$

$$\frac{1}{i\hbar}\left[\vec{P}, H\,\delta t\right] = 0 = \delta_t \vec{P}$$

$$\frac{1}{i\hbar}\left[\vec{J}, H\,\delta t\right] = 0 = \delta_t \vec{J}$$

$$\frac{1}{i\hbar}\left[\vec{N}, H\,\delta t\right] = -\vec{P}\,\delta t = \delta_t \vec{N}$$

(23) Translation in space:

$$\frac{1}{i\hbar}\left[H, \vec{P}\cdot\delta\vec{e}\right] = 0 = \delta_e H$$

$$\frac{1}{i\hbar}\left[\vec{P}, \vec{P}\cdot\delta\vec{e}\right] = 0 = \delta_e \vec{P}$$

$$\frac{1}{i\hbar}\left[\vec{J}, \vec{P}\cdot\delta\vec{e}\right] = \delta\vec{e}\times\vec{P} = \delta_e \vec{J}$$

$$\frac{1}{i\hbar}\left[\vec{N}, \vec{P}\cdot\delta\vec{e}\right] = -M\,\delta\vec{e} = \delta_e \vec{N}$$

(24) Rotations

$$\frac{1}{i\hbar}\left[H, \vec{J}\cdot\delta\vec{\omega}\right] = 0 = \delta_\omega H$$

$$\frac{1}{i\hbar}\left[\vec{P}, \vec{J}\cdot\delta\vec{\omega}\right] = \delta\vec{\omega}\times\vec{P} = \delta_\omega \vec{P}$$

$$\frac{1}{i\hbar}\left[\vec{J}, \vec{J}\cdot\delta\vec{\omega}\right] = \delta\vec{\omega}\times\vec{J} = \delta_\omega \vec{J}$$

$$\frac{1}{i\hbar}\left[\vec{N}, \vec{J}\cdot\delta\vec{\omega}\right] = \delta\vec{\omega}\times\vec{N} = \delta_\omega \vec{N}$$

(25) Boosts

$$\frac{1}{i\hbar}\left[H, \vec{N}\cdot\delta\vec{v}\right] = \vec{P}\cdot\delta\vec{v} = \delta_v H$$

$$\frac{1}{i\hbar}\left[\vec{P}, \vec{N}\cdot\delta\vec{v}\right] = M\,\delta\vec{v} = \delta_v \vec{P}$$

$$\frac{1}{i\hbar}\left[\vec{J}, \vec{N}\cdot\delta\vec{v}\right] = \delta\vec{v}\times\vec{N} = \delta_v \vec{J}$$

$$\frac{1}{i\hbar}\left[\vec{N}, \vec{N}\cdot\delta\vec{v}\right] = 0 = \delta_v \vec{N}$$