Boundary conditions on Green's function

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Hopefully the following will be suitable to be part of my forthcoming thesis. Since Green's function plays a major role in the techniques used, the following should easily fit in as a section.

This discussion was initiated to investigate the following question: How do the boundary conditions on a differential equation translate over to the respective Green's function. We claim that the Green's function equation does not make contact with the boundary conditions of the original differential equation.

A. Introduction

A non-homogeneous linear differential equation can be written in the form

$$L\,x(t) = F(t) \tag{1}$$

where L is a polynomial of order n in $\frac{d}{dt}$ with coefficients that are functions of time. $F(t)$ is a prior given function which is also called the 'source'. The corresponding homogeneous linear differential equation is

$$L\,x_0(t) = 0. \tag{2}$$

The general solution $x_0(t)$ to eq. (2) is given as a linear superposition of n independent solutions to eq. (2) and n arbitrary constants, corresponding to the n integration constants, which are fixed by the boundary conditions. If we can determine any one solution $x_1(t)$ which satisfies eq. (1) the general solution to eq. (1) is given as

$$x(t) = x_0(t) + x_1(t) \tag{3}$$

where $x_0(t)$, in this context, is called the homogeneous solution to eq. (1) and $x_1(t)$ in this situation is called the particular solution to eq. (1). One would worry if another choice of the particular solution would also lead to the same general solution. If $x_1(t)$ and $x_2(t)$ are two independent particular solutions that satisfy eq. (1), by subtracting the two equations we get

$$L\,[x_1(t) - (x_2(t)] = 0. \tag{4}$$

Thus, the arbitrariness in the general solution, $x(t)$, due to the choice of the particular solution is of the form $x_0(t)$.

Green's function method is a particular technique used for solving non-homogeneous linear differential equation. The Green's function corresponding to eq. (1) satisfies a non-homogeneous linear differential equation

$$L\,G(t,t') = \delta(t-t') \tag{5}$$

in which the source is a delta function. We shall denote the homogeneous solution to the Green's function as $G_0(t,t')$ and its particular solution as $\bar{G}(t-t')$. Any function can be written as a linear superposition of delta functions weighted by the function itself. This fact lets us write down the general solution to $x(t)$ in terms of $G(t,t')$. We shall show that the homogeneous solution $G_0(t,t')$ does not contribute to the general solution of $x(t)$. This lets us write the solution to $x(t)$ in the form

$$x(t) = \int_{-\infty}^{+\infty} dt'\bar{G}(t-t')F(t') + x_0(t) \tag{6}$$

where $x_0(t)$ is given in terms of n arbitrary constants. Thus a different particular solution $\bar{G}'(t-t')$ will be off from $\bar{G}(t-t')$ by a homogeneous form, and since the homogeneous solution $G_0(t,t')$ does not contribute to $x(t)$, any one particular solution to the Green's function is enough to write eq. (6).

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B. Differential equation

We shall consider the differential equation satisfied by $x(t)$

$$-\left(\frac{d^2}{dt^2}+\omega^2\right)x(t)=F(t) \tag{7}$$

where $F(t)$ is a prior given source, and the initial conditions are, $x(0)=-A$, and $\dot{x}(0)=0$.

C. The corresponding Green's function equation

The corresponding differential equation for the Green's function, $G(t,t')$, is

$$-\left(\frac{d^2}{dt^2}+\omega^2\right)G(t,t')=\delta(t-t'). \tag{8}$$

Integrating eq. (8) over an infinitesimal region around the point t' lets us extract the information pertaining to the continuity of the derivative of the Green's function at $t=t'$. This reads as

$$-\left\{\frac{d}{dt}G(t-t')\right\}\bigg|_{t=t'+\epsilon}+\left\{\frac{d}{dt}G(t-t')\right\}\bigg|_{t=t'-\epsilon}=1. \tag{9}$$

We also observe that using equations (8) and (9) we can learn about the continuity of the Green's function at $t=t'$. In this regard we note that Eq. (9) can be satisfied with either the Green's function being finitely discontinous at $t=t'$, or it being continous at the point $t=t'$. In particular we could ask, can equations (8) and (9) accomodate a finite discontinuity in the Green's function? With a finite discontinuity in the Green's function at $t=t'$ the $\frac{d^2}{dt^2}G(t,t')$ term in eq. (8) will contribute a term involving the derivative of a delta function, which is not the required quantity on the right hand side of the equation. On the contrary, with a Green's function continous at $t=t'$ the $\frac{d^2}{dt^2}G(t,t')$ term in eq. (8) will contribute a delta function, as required to balance the right hand side. Thus we conclude

$$-G(t-t')|_{t=t'+\epsilon}+G(t-t')|_{t=t'-\epsilon}=0. \tag{10}$$

Thus, eq. (8), the differential equation for the Green's function, inherently enforces two continuity conditions, given in equations (9) and (10), on the Green's function at $t=t'$.

The homogeneous differential equation corresponding to Eq. (8) is

$$-\left(\frac{d^2}{dt^2}+\omega^2\right)G_0(t,t')=0 \tag{11}$$

where $G_0(t,t')$ is the corresponding homogeneous solution. The part of the Green's function that satisfies eq. (8) such as to privide the delta function term on the right hand side is called the particular solution. Thus in general we can write

$$G(t,t')=G_0(t,t')+\bar{G}(t-t') \tag{12}$$

where $\bar{G}(t-t')$ is the particular solution. The homogeneous solution does not make any contact with the point t', and thus we do not require its functional dependence to be of the form $t-t'$. On the contrary, due to the presence of the delta function in eq. (8), translational invariance requires the functional dependence of $\bar{G}$ to be of the form $t-t'$. Before imposing any boundary conditions, both these solutions, $G_0(t,t')$ and $\bar{G}(t-t')$, will be expressed in terms of two arbitrary constants each. Also, difference between any two possible particular solutions will be a homogeneous solution. Thus, in principle, any one choice of the particular solution summed to the homogeneous solution which comes with two arbitrary constants defines a general solution to eq. (8).

D. Solution to $x(t)$ in terms of $G(t,t')$

We substitute $t\to t'$ in eq. (7), multiply it by $G(t-t')$ and integrate it with respect to t'; independently, we multiply eq. (8) with $x(t')$ and integrate it with respect to t'; and subsequently subtract the modified equations to get

$$x(t)=x_{G_0}(t)+\bar{x}(t) \tag{13}$$

where the contribution to $x(t)$ from the homogeneous part of the Green's function is captured in

$$x_{G_0}(t) = \int_{-\infty}^{+\infty} dt' G_0(t,t') F(t') + \int_{-\infty}^{+\infty} dt' G_0(t,t') \frac{d^2}{dt'^2} x(t') - \int_{-\infty}^{+\infty} dt' x(t') \frac{d^2}{dt'^2} G_0(t,t') \tag{14}$$

which evaluates to zero when we use equations (7) and (11). Thus the homogeneous part of the Green's function does not contribute to $x(t)$. The second term in eq. (13),

$$\begin{aligned} \bar{x}(t) &= \int_{-\infty}^{+\infty} dt' \bar{G}(t,t') F(t') + \int_{-\infty}^{+\infty} dt' \frac{d}{dt'} \left[\bar{G}(t-t') \dot{x}(t') - x(t') \frac{d}{dt'} \bar{G}(t-t') \right] \\ &= \int_{-\infty}^{+\infty} dt' \bar{G}(t-t') F(t') + \lim_{\tau_2 \to +\infty} \left(\dot{x}(\tau_2) - x(\tau_2) \frac{d}{d\tau_2} \right) \bar{G}(t-\tau_2) - \lim_{\tau_1 \to -\infty} \left(\dot{x}(\tau_1) - x(\tau_1) \frac{d}{d\tau_1} \right) \bar{G}(t-\tau_1), \end{aligned} \tag{15}$$

where we isolated the surface terms using integration by parts. The limiting variables are assumed to satisfy $\tau_1 < \{t, t'\} < \tau_2$. We note that the surface terms satisfy the homegeneous equation corresponding to eq. (7)

$$-\left(\frac{d^2}{dt^2} + \omega^2 \right) \left(\dot{x}(\tau) - x(\tau) \frac{d}{d\tau} \right) \bar{G}(t-\tau) = 0 \tag{16}$$

because the surface points, denoted by τ, never equals the variable t, i.e. $\tau \neq t$. Since the homogeneous equation corresponding to eq. (7) has oscillatory solutions we can write

$$x(t) = \int_{-\infty}^{+\infty} dt' \bar{G}(t-t') F(t') + \alpha_0 \, e^{i\omega t} + \beta_0 \, e^{-i\omega t} \tag{17}$$

where α_0 and β_0 are the arbitrary numerical constants. The boundary conditions to eq. (7) takes away the arbitrariness from α_0 and β_0.

In summary, the homogeneous part of the Green's function does not contribute to $x(t)$. The homogeneous part of $x(t)$ gets its contribution from, $x(\tau)$ and $\bar{G}(t-\tau)$, where τ is evaluated at the surface. We shall look into the structure of α_0 and β_0 in terms of the surface points in more detail in section G.

E. Solving for the Green's function without imposing boundary conditions

With the above observations in mind we proceed to solve for the Green's function. For all points, except $t = t'$, the differential eq. (8) has no source term and thus reads like the eq. for $G_0(t,t')$ in eq. (11). This eq. has oscillatory solutions, which could have different behaviour at $t < t'$ and $t > t'$, except for the constraint imposed by the continuity conditions in eqs. (9) and (10). In terms of four arbitray functions of t', A, B, C, and D, we can write

$$G(t,t') = \begin{cases} A(t') \, e^{i\omega t} + B(t') \, e^{-i\omega t} & \text{if } t < t', \\ C(t') \, e^{i\omega t} + D(t') \, e^{-i\omega t} & \text{if } t > t'. \end{cases} \tag{18}$$

Imposing the continuity conditions in eqs. (9) and (10) we get the following equations constraining $A(t')$, $B(t')$, $C(t')$, and $D(t')$:

$$[C(t') - A(t')] \, e^{i\omega t'} + [D(t') - B(t')] \, e^{-i\omega t'} = 0 \tag{19}$$

$$[C(t') - A(t')] \, e^{i\omega t'} + [D(t') - B(t')] \, e^{-i\omega t'} = \frac{i}{\omega}. \tag{20}$$

From the structure of the above the constraints we observe that the equations let us solve for in combinations $A \,\&\, C$, and $B \,\&\, D$. This lets us solve for $G(t,t')$ in the following four forms:

$$\begin{aligned} G(t,t') &= A(t') \, e^{i\omega t} + B(t') \, e^{-i\omega t} + \bar{G}_R(t-t') & \text{(21a)} \\ &= C(t') \, e^{i\omega t} + D(t') \, e^{-i\omega t} + \bar{G}_A(t-t') & \text{(21b)} \\ &= A(t') \, e^{i\omega t} + D(t') \, e^{-i\omega t} + \bar{G}_F(t-t') & \text{(21c)} \\ &= C(t') \, e^{i\omega t} + B(t') \, e^{-i\omega t} + \bar{G}_W(t-t') & \text{(21d)} \end{aligned}$$

where

$$\bar{G}_R(t-t') = -\frac{1}{\omega}\frac{1}{2i}\left[+\theta(t-t')\,e^{i\omega(t-t')} - \theta(t-t')\,e^{-i\omega(t-t')}\right] = -\frac{1}{\omega}\,\theta(t-t')\,\sin\omega(t-t') \tag{22a}$$

$$\bar{G}_A(t-t') = -\frac{1}{\omega}\frac{1}{2i}\left[-\theta(t'-t)\,e^{i\omega(t-t')} + \theta(t'-t)\,e^{-i\omega(t-t')}\right] = +\frac{1}{\omega}\,\theta(t'-t)\,\sin\omega(t-t') \tag{22b}$$

$$\bar{G}_F(t-t') = -\frac{1}{\omega}\frac{1}{2i}\left[+\theta(t-t')\,e^{i\omega(t-t')} + \theta(t'-t)\,e^{-i\omega(t-t')}\right] \tag{22c}$$

$$\bar{G}_W(t-t') = -\frac{1}{\omega}\frac{1}{2i}\left[-\theta(t'-t)\,e^{i\omega(t-t')} - \theta(t-t')\,e^{-i\omega(t-t')}\right] \tag{22d}$$

and the subscripts stand for retarded, advanced, Fynmann, and Wheeler, respectively. It is easy to recognize that the above four forms are special cases of the following general expression for $\bar{G}(t-t')$

$$\bar{G}(t-t') = -\frac{1}{\omega}\frac{1}{2i}\left[\,a\,\theta(t-t') - b\,\theta(t'-t)\right]e^{i\omega(t-t')} + \frac{1}{\omega}\frac{1}{2i}\left[\,c\,\theta(t-t') - d\,\theta(t'-t)\right]e^{-i\omega(t-t')} \tag{23}$$

where the numerical constants a, b, c, and d, are arbitrary to the extent that they obey the constraints $a+b=1$, and $c+d=1$. The special cases, $a=1$, $c=1$, corresponds to $\bar{G}_R$; $a=0$, $c=0$, corresponds to $\bar{G}_A$; $a=1$, $c=0$, corresponds to $\bar{G}_F$; and $a=0$, $c=1$, corresponds to $\bar{G}_W$, respectively. As we mentioned earlier the particular Green's function $\bar{G}(t-t')$ is not unique due to the arbitrariness in the choice of a and c. We also said that the difference between any two particular solutions will be a homogeneous solution. This can be illustrated by making finite changes in a and c given as, $a' = a+\delta_1$, and $c' = c+\delta_2$, and studying for the variation in $\bar{G}(t-t')$ due to this change. We get

$$\Delta\bar{G}(t-t') = -\frac{1}{\omega}\frac{1}{2i}\left[\,\delta_1\,\theta(t-t') + \delta_1\,\theta(t'-t)\right]e^{i\omega(t-t')} + \frac{1}{\omega}\frac{1}{2i}\left[\,\delta_2\,\theta(t-t') + \delta_2\,\theta(t'-t)\right]e^{-i\omega(t-t')} \tag{24}$$

$$= -\frac{1}{\omega}\frac{1}{2i}e^{i\omega(t-t')} + \frac{1}{\omega}\frac{1}{2i}e^{-i\omega(t-t')} \tag{25}$$

which indeed can be absorbed into the homegeneous part of the solution.

F. Solution to $x(t)$

Subjecting eq. (17), in conjunction with eq. (23), to the initial conditions $x(0) = -A$ and $\dot{x}(0) = 0$ we get

$$\alpha_0 + \beta_0 = -A + \frac{1}{2i\omega}\int_{-\infty}^{0} dt' F(t')\left[a\,e^{-i\omega t'} - c\,e^{i\omega t'}\right] - \frac{1}{2i\omega}\int_{0}^{+\infty} dt' F(t')\left[b\,e^{-i\omega t'} - d\,e^{i\omega t'}\right] \tag{26}$$

$$\alpha_0 - \beta_0 = -0 + \frac{1}{2i\omega}\int_{-\infty}^{0} dt' F(t')\left[a\,e^{-i\omega t'} + c\,e^{i\omega t'}\right] - \frac{1}{2i\omega}\int_{0}^{+\infty} dt' F(t')\left[b\,e^{-i\omega t'} + d\,e^{i\omega t'}\right] \tag{27}$$

where the differentiations with respect to the limits on the integrals contributes zero because they total up to

$$\frac{1}{\omega}\frac{1}{2i}F(0)\left[(a+b)-(c+d)\right] = 0 \tag{28}$$

because $a+b = c+d = 1$. Solving for α_0 and β_0, in the above equations, gives us

$$\alpha_0 = -\frac{A}{2} + a\,\frac{1}{\omega}\frac{1}{2i}\int_{-\infty}^{0} dt' F(t')e^{-i\omega t'} - b\,\frac{1}{\omega}\frac{1}{2i}\int_{0}^{+\infty} dt' F(t')e^{-i\omega t'} \tag{29a}$$

$$\beta_0 = -\frac{A}{2} - c\,\frac{1}{\omega}\frac{1}{2i}\int_{-\infty}^{0} dt' F(t')e^{+i\omega t'} + d\,\frac{1}{\omega}\frac{1}{2i}\int_{0}^{+\infty} dt' F(t')e^{+i\omega t'}. \tag{29b}$$

Using the above expressions for α_0 and β_0 in eq. (17) we get

$$x(t) = -A\cos\omega t - (a+b)\frac{1}{2i\omega}\int_0^t dt' F(t')e^{i\omega(t-t')} + (c+d)\frac{1}{2i\omega}\int_0^t dt' F(t')e^{-i\omega(t-t')} \tag{30}$$

$$= -A\cos\omega t - \frac{1}{\omega}\int_0^t dt' F(t')\sin\omega(t-t') \tag{31}$$

where we used $a+b=1$ and $c+d=1$. Differentiating the above expression with t gives us

$$\dot{x}(t) = +\omega A\sin\omega t - \int_0^t dt' F(t')\cos\omega(t-t') \tag{32}$$

where the differentiation with respect to the integration limit in eq. (31) contributed zero.

G. Consistency check

Let us evaluate α_0 and β_0 using a different route. Let us get back to eq. (15). Let us define the functions

$$\begin{aligned} S_\pm(\tau) &= \frac{1}{\omega}\frac{1}{2i}\left(\dot{x}(\tau) - x(\tau)\frac{d}{d\tau}\right)e^{\pm i\omega\tau} \\ &= \pm\frac{A}{2} - \frac{1}{\omega}\frac{1}{2i}\int_0^\tau dt' F(t')e^{\pm i\omega t'} \end{aligned} \tag{33}$$

where we used equations (31) and (32) for the evaluation. In terms of the finctions $S_\pm(\tau)$ we can write the surface terms in eq. (15) as

$$\lim_{\tau_2\to+\infty}\left(\dot{x}(\tau_2) - x(\tau_2)\frac{d}{d\tau_2}\right)\bar{G}(t-\tau_2) = +b\,S_-(+\infty)e^{+i\omega t} - d\,S_+(+\infty)e^{-i\omega t} \tag{34a}$$

$$\lim_{\tau_1\to-\infty}\left(\dot{x}(\tau_1) - x(\tau_1)\frac{d}{d\tau_1}\right)\bar{G}(t-\tau_1) = -a\,S_-(-\infty)e^{+i\omega t} + c\,S_+(-\infty)e^{-i\omega t}. \tag{34b}$$

Using the above expressions for the surface terms in eq. (15) in conjunction with eq. (17) gives us

$$\begin{aligned} \alpha_0 &= +a\,S_-(-\infty) + b\,S_-(+\infty) \\ &= -\frac{A}{2} + a\frac{1}{\omega}\frac{1}{2i}\int_{-\infty}^0 dt' F(t')e^{-i\omega t'} - b\frac{1}{\omega}\frac{1}{2i}\int_0^{+\infty} dt' F(t')e^{-i\omega t'} \end{aligned} \tag{35a}$$

$$\begin{aligned} \beta_0 &= -c\,S_+(-\infty) - d\,S_+(+\infty) \\ &= -\frac{A}{2} - c\frac{1}{\omega}\frac{1}{2i}\int_{-\infty}^0 dt' F(t')e^{+i\omega t'} + d\frac{1}{\omega}\frac{1}{2i}\int_0^{+\infty} dt' F(t')e^{+i\omega t'} \end{aligned} \tag{35b}$$

which is exactly the expressions we got earlier in eq. (29).

APPENDIX A

1. Contour integral representation of eq.(23)

We shall begin by noting the following integral representations:

$$\lim_{\epsilon\to0+}\int_{-\infty}^{+\infty} d\omega\,\frac{e^{i\omega x}}{\omega-(a+i\epsilon)} = +2\pi i\,\theta(x)e^{iax} \tag{A1}$$

$$\lim_{\epsilon\to0+}\int_{-\infty}^{+\infty} d\omega\,\frac{e^{i\omega x}}{\omega-(a-i\epsilon)} = -2\pi i\,\theta(-x)e^{iax} \tag{A2}$$