SET - 17

Modified Bessel functions

(1) For a planar geometry we have

$$G(\vec{r},\vec{r}') = \frac{1}{\epsilon_0}\int \frac{d^2k_\perp}{(2\pi)^2}\, e^{i\vec{k}_\perp\cdot(\vec{r}-\vec{r}')_\perp}\, \frac{1}{2k_\perp}\, e^{-k_\perp|z-z'|} \qquad \vec{P} = (\vec{r}-\vec{r}')_\perp$$

$$= \frac{1}{4\pi\epsilon_0}\int_0^\infty dk_\perp\, J_0(k_\perp P)\, e^{-k_\perp|z-z'|}$$

(2) Observe the integral

$$\int_{-\infty}^{+\infty} \frac{dk_z}{2\pi}\, \frac{e^{ik_z(z-z')}}{k_z^2+k_\perp^2} = \frac{1}{2k_\perp}\, e^{-k_\perp|z-z'|}$$

→ use contour integrals to evaluate this — HW.

(3) Using (2) in (1)

$$G(\vec{r},\vec{r}') = \frac{1}{2\pi\epsilon_0}\int_0^\infty k_\perp\, dk_\perp\, J_0(k_\perp P)\int_{-\infty}^{+\infty}\frac{dk_z}{2\pi}\,\frac{e^{ik_z(z-z')}}{k_z^2+k_\perp^2}$$

$$= \frac{1}{2\pi\epsilon_0}\int_{-\infty}^{+\infty}\frac{dk_z}{2\pi}\, e^{ik_z(z-z')}\, \underbrace{\boxed{\int_0^\infty k_\perp\, dk_\perp\, \frac{J_0(k_\perp P)}{k_\perp^2+k_z^2}}}_{K_0(k_z P)}$$

(4) Modified Bessel function is defined as

$$K_0(t) = \int_0^\infty s\, ds\, \frac{J_0(s)}{s^2+t^2} \qquad 0<t<\infty$$

(5) Thus, we have.

$$G(\vec{r},\vec{r}') = \frac{1}{4\pi\epsilon_0}\int_0^\infty dk_\perp \, J_0(k_\perp P)\, e^{-k_\perp|z-z'|} \quad \rightarrow \text{planar geometry}$$

$$= \frac{1}{2\pi\epsilon_0}\int_{-\infty}^{\infty} \frac{dk_z}{2\pi}\, e^{ik_z(z-z')}\, K_0(k_z P) \quad \rightarrow \text{cylindrical geometry}$$

(6) $$\frac{1}{2\pi\epsilon_0}\int_{-\infty}^{0} \frac{dk_z}{2\pi}\, e^{ik_z(z-z')}\, K_0(k_z P) = \frac{1}{2\pi\epsilon_0}\int_{\infty}^{0} -\frac{dk_z'}{2\pi}\, e^{-ik_z'(z-z')}\, K_0(-k_z' P) \qquad k_z = -k_z'$$

$$= \frac{1}{2\pi\epsilon_0}\int_0^\infty \frac{dk_z}{2\pi}\, e^{-ik_z(z-z')}\, K_0(k_z P)$$

Thus we have

$$G(\vec{r},\vec{r}') = \frac{1}{2\pi\epsilon_0}\int_0^\infty \frac{dk_z}{2\pi}\left[e^{ik_z(z-z')} + e^{-ik_z(z-z')}\right] K_0(k_z P)$$

$$= \frac{1}{4\pi\epsilon_0}\,\frac{2}{\pi}\int_0^\infty dk_z \, \cos\left[k_z(z-z')\right] K_0(k_z P)$$

which brings out the oscillatory nature in the z-direction more explicitly.

(7)
$$K_0(k_z P) = \int_0^\infty k_\perp \, dk_\perp \, \frac{J_0(k_\perp P)}{k_\perp^2 + k_z^2}$$

$$= \int_0^\infty k_\perp \, dk_\perp \int_0^{2\pi} \frac{d\alpha}{2\pi} \, e^{i k_\perp P \cos\alpha} \, \frac{1}{k_\perp^2 + k_z^2}$$

$$= 2\pi \int \frac{d^2 k_\perp}{(2\pi)^2} \, e^{i \vec{k}_\perp \cdot (\vec{r} - \vec{r}')_\perp} \, \frac{1}{k_\perp^2 + k_z^2}$$

(8) The above form immediately suggests

$$\left(-\nabla_\perp^2 + k_z^2\right) \frac{1}{2\pi} K_0(k_z P)$$

$$= \int \frac{d^2 k_\perp}{(2\pi)^2} \left(-\nabla_\perp^2 + k_z^2\right) e^{i \vec{k}_\perp \cdot (\vec{r} - \vec{r}')_\perp} \, \frac{1}{k_\perp^2 + k_z^2}$$

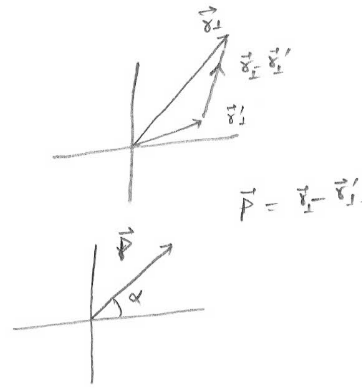
$$= \int \frac{d^2 k_\perp}{(2\pi)^2} \left(-(i k_\perp)^2 + k_z^2\right) e^{i \vec{k}_\perp \cdot (\vec{r} - \vec{r}')_\perp} \, \frac{1}{k_\perp^2 + k_z^2}$$

$$= \int \frac{d^2 k_\perp}{(2\pi)^2} \, e^{i \vec{k}_\perp \cdot (\vec{r} - \vec{r}')_\perp}$$

$$= \delta^{(2)}(\vec{r}_\perp - \vec{r}'_\perp)$$

Thus, we have the Green's function equation

$$\left[-\nabla_\perp^2 + k_z^2\right] \frac{1}{2\pi} K_0(k_z P) = \delta^{(2)}(\vec{r}_\perp - \vec{r}'_\perp)$$



(9) Using

$$\delta^{(2)}(\vec{r}_\perp - \vec{r}_\perp') = \frac{\delta(\rho - \rho')}{\rho}\,\delta(\phi - \phi')$$

$$\vec{r}_\perp = (\rho \cos\phi,\ \rho \sin\phi)$$

$$r_\perp' = (\rho' \cos\phi',\ \rho' \sin\phi')$$

$$\vec{P} = (\rho \cos\phi - \rho' \cos\phi',\ \rho \sin\phi - \rho' \sin\phi')$$

$$P = \sqrt{\rho^2 + \rho'^2 - 2\rho\rho' \cos(\phi - \phi')}$$

$$\tan\alpha = \frac{\rho \sin\phi - \rho' \sin\phi'}{\rho \cos\phi - \rho' \cos\phi'}.$$

In the limit $\vec{r}_\perp \to \vec{r}_\perp'$, $\rho \to \rho'$

$$P \to \rho\sqrt{2 - 2\cos(\phi - \phi')} = 2\rho \sin\left(\frac{\phi - \phi'}{2}\right) \to 0$$

$$\tan\alpha \to -\frac{\rho(\phi - \phi')}{\rho\left(\frac{\phi^2}{2} - \frac{\phi'^2}{2}\right)} = -\frac{2}{\phi + \phi'} = -\frac{1}{\phi} \quad ?$$

Thus, argue that, which is intuitively clear

$$\delta^{(2)}(\vec{r}_\perp - \vec{r}_\perp') = \frac{\delta(\rho - \rho')}{\rho}\,\delta(\phi - \phi') = \frac{\delta(P)}{P}\,\frac{1}{2\pi}$$

10 Using 9 in 8 we have.

$$\left[-\nabla_\perp^2 + k_z^2\right] \frac{1}{2\pi} K_0(k_z \rho) = \frac{\delta(\rho)}{\rho} \frac{1}{2\pi}$$

$$\left[-\frac{1}{\rho}\frac{d}{d\rho}\rho\frac{d}{d\rho} + k_z^2\right] K_0(k_z \rho) = \frac{\delta(\rho)}{\rho}$$

$$\left[-\frac{1}{t}\frac{d}{dt}t\frac{d}{dt} + 1\right] K_0(t) = \frac{\delta(t)}{t}$$

$$\left[-\frac{1}{t}\frac{d}{dt}t\frac{d}{dt} + 1\right] K_0(t) = 0 \qquad t \neq 0.$$

11 Replacing $t \to it$

$$\left[\frac{1}{t}\frac{d}{dt}t\frac{d}{dt} + 1\right] K_0(it) = 0$$

which reveals that

$$J_0(t) = K_0(it).$$

12
$$\left[-\frac{1}{\rho}\frac{d}{d\rho}\rho\frac{d}{d\rho} + k_z^2\right] K_0(k_z \rho) = \left[-\frac{1}{\rho}\frac{d}{d\rho}\rho\frac{d}{d\rho} + k_z^2\right] \int_0^\infty k_\perp dk_\perp \frac{J_0(k_\perp \rho)}{k_\perp^2 + k_z^2}.$$

$$= \int_0^\infty k_\perp dk_\perp \left[-\frac{1}{\rho}\frac{d}{d\rho}\rho\frac{d}{d\rho} + k_z^2\right] J_0(k_\perp \rho) \frac{1}{k_\perp^2 + k_z^2}$$

$$= \int_0^\infty k_\perp dk_\perp J_0(k_\perp \rho) \qquad \int_0^\infty k_\perp dk_\perp J_0(k_\perp \rho) J_0(k_\perp \rho') = \frac{\delta(\rho-\rho')}{\rho}$$

$$= \frac{\delta(\rho)}{\rho}$$

(13)
$$\left[\underbrace{-\frac{1}{\rho}\frac{d}{d\rho}\rho\frac{d}{d\rho}}_{-\nabla_\perp^2} + k_z^2\right] K_0(k_z\rho) = \frac{\delta(\rho)}{\rho}$$

Using divergence theorem in two dimension.

$$-\int d\vec{s}\cdot\vec{\nabla}_\perp K_0(k_\perp\rho) = 2\pi$$

circle.

$$-2\pi\rho\frac{d}{d\rho}K_0(k_\perp\rho) = 2\pi$$

$$\frac{d}{dt}K_0(t) = -\frac{1}{t}$$

$$K_0(t) = -\ln t + \text{const}$$

$$= \ln\frac{2}{t} - \gamma$$

Euler's constant
$\gamma = 0.577$