Macroscopic energy and momentum

1. We earlier derived the statement of conservation of energy and conservation of momentum for

$$\vec{D} = \epsilon_0 \vec{E}$$
$$\vec{B} = \mu_0 \vec{H}.$$

We would now like to look into the conservation statements for

$$\vec{D} = \epsilon_0 \vec{E} + \vec{P} = \overleftrightarrow{\epsilon} \cdot \vec{E}$$
$$\vec{B} = \mu_0 \vec{H} + \mu_0 \vec{M} = \overleftrightarrow{\mu} \cdot \vec{H}$$

2. The macroscopic Maxwell's equations are.

$$\vec{\nabla} \cdot \vec{D} = \rho_e$$
$$\vec{\nabla} \cdot \vec{B} = 0 = \rho_m$$
$$-\vec{\nabla} \times \vec{E} - \frac{\partial \vec{B}}{\partial t} = 0 = \vec{J}_m$$
$$\vec{\nabla} \times \vec{H} - \frac{\partial \vec{D}}{\partial t} = \vec{J}_e.$$

$$\vec{D} = \epsilon_0 \vec{E} + \vec{P} = \overleftrightarrow{\epsilon} \cdot \vec{E}$$
$$\vec{B} = \mu_0 \vec{H} + \mu_0 \vec{M} = \overleftrightarrow{\mu} \cdot \vec{H}$$

$$\vec{F} = q_e \vec{E} + q_e \vec{V}_e \times \vec{B} + q_m \vec{H} - q_m \vec{V}_m \times \vec{D}.$$

(3)
$$\vec{j}_e \cdot \vec{E} = \left[\vec{\nabla}\times\vec{H} - \frac{\partial \vec{D}}{\partial t}\right]\cdot\vec{E}$$
$$\vec{j}_m \cdot \vec{H} = 0 = \left[-\vec{\nabla}\times\vec{E} - \frac{\partial \vec{B}}{\partial t}\right]\cdot\vec{H}$$

(4) Adding the above equations we have.
$$\vec{j}_e\cdot\vec{E} = (\vec{\nabla}\times\vec{H})\cdot\vec{E} - (\vec{\nabla}\times\vec{E})\cdot\vec{H} - \vec{E}\cdot\frac{\partial}{\partial t}\vec{D} - \vec{H}\cdot\frac{\partial \vec{B}}{\partial t}$$
$$= -\vec{\nabla}\cdot\underbrace{(\vec{E}\times\vec{H})}_{\vec{S}} - \frac{\partial}{\partial t}U$$

which requires.
$$\frac{\partial U}{\partial t} \overset{?}{=} \vec{E}\cdot\frac{\partial}{\partial t}\vec{D} + \vec{H}\cdot\frac{\partial}{\partial t}\vec{B}$$

(5) Similarly,
$$\vec{f} = \rho_e\vec{E} + \vec{j}_e\times\vec{B}$$
$$\rho_e\vec{E} + \vec{j}_e\times\vec{B} = (\vec{\nabla}\cdot\vec{D})\vec{E} + \left[\vec{\nabla}\times\vec{H} - \frac{\partial\vec{D}}{\partial t}\right]\times\vec{B}$$
$$\rho_m\vec{H} - \vec{j}_m\times\vec{D} = 0 = (\vec{\nabla}\cdot\vec{B})\vec{H} + \left[\vec{\nabla}\times\vec{E} + \frac{\partial\vec{B}}{\partial t}\right]\times\vec{D}$$

(6) Adding the above equations leads to
$$\vec{f} = (\vec{\nabla}\cdot\vec{D})\vec{E} + (\vec{\nabla}\times\vec{E})\times\vec{D} + (\vec{\nabla}\cdot\vec{B})\vec{H} + (\vec{\nabla}\times\vec{H})\times\vec{B} - \frac{\partial}{\partial t}(\vec{D}\times\vec{B})$$
$$= -D_i\vec{\nabla}E_i + \vec{\nabla}\cdot(\vec{D}\vec{E}) - B_i\vec{\nabla}H_i + \vec{\nabla}\cdot(\vec{B}\vec{H}) - \frac{\partial}{\partial t}(\vec{D}\times\vec{B})$$
$$= -\vec{\nabla}\cdot\overleftrightarrow{T} - \frac{\partial}{\partial t}\underbrace{(\vec{D}\times\vec{B})}_{\vec{G}}$$

which requires.
$$\vec{\nabla}\cdot\overleftrightarrow{T} = D_i\vec{\nabla}E_i - \vec{\nabla}\cdot(\vec{D}\vec{E}) + B_i\vec{\nabla}H_i - \vec{\nabla}\cdot(\vec{B}\vec{H})$$

7 Using 3 & 6 we have.

$$\vec{j_e} \cdot \vec{E} + \vec{\nabla} \cdot \vec{S} + \frac{\partial}{\partial t} U = 0$$

where $\vec{S} = \vec{E} \times \vec{H}$

$$\frac{\partial U}{\partial t} \stackrel{?}{=} \vec{E} \cdot \frac{\partial}{\partial t} \vec{D} + \vec{H} \cdot \frac{\partial}{\partial t} \vec{B}$$

and

$$\vec{f} + \vec{\nabla} \cdot \overleftrightarrow{T} + \frac{\partial}{\partial t} \vec{G} = 0$$

where $\vec{G} = \vec{D} \times \vec{B}$

$$\vec{\nabla} \cdot \overleftrightarrow{T} \stackrel{?}{=} D_i \vec{\nabla} E_i - \vec{\nabla} \cdot (\vec{D} \vec{E}) + B_i \vec{\nabla} H_i - \vec{\nabla} \cdot (\vec{B} \vec{H})$$

8 Non dispersive medium. is described by.

$$\vec{D} = \epsilon \vec{E} \quad \text{and} \quad \vec{B} = \mu \vec{H}$$

where ϵ and μ are constants will respect to x and t, i.e., it is homogeneous, isotropic ($\overleftrightarrow{\epsilon} = \epsilon \overleftrightarrow{1}$). and nondispersive.

$$\vec{E} \cdot \frac{\partial}{\partial t} \vec{D} + \vec{H} \cdot \frac{\partial}{\partial t} \vec{B} = \frac{\partial}{\partial t} \underbrace{\left[\frac{1}{2} \epsilon \vec{E} \cdot \vec{E} + \frac{1}{2} \mu \vec{H} \cdot \vec{H} \right]}_{U}$$

and

$$D_i \vec{\nabla} E_i - \vec{\nabla} \cdot (\vec{D} \vec{E}) + B_i \vec{\nabla} H_i - \vec{\nabla} \cdot (\vec{B} \vec{H})$$

$$= \vec{\nabla} \cdot \underbrace{\left[\frac{1}{2} \epsilon E^2 + \frac{1}{2} \mu H^2 - \epsilon \vec{E} \vec{E} - \mu \vec{H} \vec{H} \right]}_{\overleftrightarrow{T}}$$

9 We shall now define the velocity of center of energy. Consider a region with $\rho_e = 0$, $\vec{J}_e = 0$,

$$\vec{r} \left[\frac{\partial}{\partial t} U + \vec{\nabla} \cdot \vec{S} \right] = 0$$

$$\int d^3r \; \vec{r} \left[\frac{\partial}{\partial t} U + \vec{\nabla} \cdot \vec{S} \right] = 0$$

$$\frac{\partial}{\partial t} \left(\int d^3r \; \vec{r} \; U \right) + \int d^3r \; \vec{\nabla} \cdot (\vec{S} \; \vec{r}) - \int d^3r \; \vec{S} = 0$$

10 Since an electromagnetic pulse has a finite extent, we can choose the volume such that the surface term does not contribute.

Thus,

$$\frac{\partial}{\partial t} \left(\int d^3r \; \vec{r} \; U \right) = \int d^3r \; \vec{S}$$

11 Define.

$$\langle \vec{r} \rangle_E = \frac{\int d^3r \; \vec{r} \; U}{\int d^3r \; U} \longrightarrow \text{centroid of energy}$$

$$\vec{V}_E = \frac{d}{dt} \langle \vec{r} \rangle_E \longrightarrow \text{velocity of centroid of energy.}$$

12

$$\vec{V}_E = \frac{\left(\int d^3r \; \vec{S} \right)}{\left(\int d^3r \; U \right)}$$

13 For a nondispersive medium we obtained (in 8)

$$U = \frac{1}{2}\epsilon E^2 + \frac{1}{2}\mu H^2.$$

$$\vec{S} = \vec{E}\times\vec{H}$$

14 For a monochromatic wave, using Maxwell's equation in 2,

$$\vec{\nabla}\times(\vec{\nabla}\times\vec{H}) = \frac{\partial}{\partial t}\vec{\nabla}\times\vec{D}$$

$$-\nabla^2\vec{H} = \epsilon\mu\frac{\partial}{\partial t}\vec{H},$$

we can show

$$\epsilon E^2 = \mu H^2 \quad \text{and} \quad \vec{E}\cdot\vec{H} = 0$$

15 Using 13 and 14 in 12

$$|\vec{V}_E| = \frac{\int d^3r\ E H}{\int d^3r\ \epsilon E^2} = \frac{EH}{\epsilon E^2}$$

$|\vec{S}| = |\vec{E}||\vec{H}| = EH$
($\because \vec{E}\cdot\vec{H} = 0$)

$$V_E = \frac{1}{\sqrt{\epsilon\mu}}$$

$$\frac{1}{n} = \frac{V_E}{c} = \sqrt{\frac{\epsilon_0\mu_0}{\epsilon\mu}}$$

refractive index.

16 Let us now consider a <u>dispersive medium</u> described by

$$\omega_p^2 = \frac{ne^2}{m\epsilon_0}$$

$$\frac{\epsilon(\omega)}{\epsilon_0} = 1 + \frac{\omega_p^2}{\omega_0^2 - \omega^2 - i\omega\gamma}$$

17 For large $\omega >> \omega_0$, we have.

$$\frac{\epsilon(\omega)}{\epsilon_0} = 1 - \frac{\omega_p^2}{\omega^2},$$

which would seem to suggest, using 15, naively,

$$\frac{1}{n} = \frac{V_E}{c} > 1,$$

which would violate causality.

18 To correct this erroneous conclusion, we need to return to 7,

$$\frac{\partial}{\partial t} U(t) \stackrel{?}{=} \vec{E}(t) \cdot \frac{\partial}{\partial t} \vec{D}(t) + \vec{H}(t) \cdot \frac{\partial}{\partial t} \vec{B}(t),$$

to identify the appropriate $U(t)$.

(19) We shall use

(i) $\vec{E}(t)$ is real:

$$\vec{E}(t) = \vec{E}(t)^* \quad \Rightarrow \quad \vec{E}(-\omega)^* = \vec{E}(\omega)$$

and

$$\vec{E}(t) = \int_{-\infty}^{+\infty} \frac{d\omega}{2\pi}\, e^{-i\omega t}\, \vec{E}(\omega) = \int_{-\infty}^{+\infty} \frac{d\omega}{2\pi}\, e^{i\omega t}\, \vec{E}(-\omega)$$

(ii) $\vec{D}(t)$ is real:

$$\overleftrightarrow{\epsilon}(-\omega)^* = \overleftrightarrow{\epsilon}(\omega),$$

and, further, for simplicity in analysis, we shall assume

$\overleftrightarrow{\epsilon}(\omega)$ is real (no absorption).

$$\overleftrightarrow{\epsilon}(-\omega) = \overleftrightarrow{\epsilon}(\omega).$$

(iii) $\overleftrightarrow{\epsilon}(\omega)$ by construction is necessarily symmetric,

$$\epsilon_{ij}(\omega) = \epsilon_{ji}(\omega).$$

(20) $$\vec{E}(t)\,\frac{\partial}{\partial t}\,\vec{D}(t) = \frac{1}{2}\,E(t)^{*}\cdot\frac{\partial}{\partial t}\,\vec{D}(t) + \frac{1}{2}\,\vec{E}(t)\cdot\frac{\partial}{\partial t}\,\vec{D}(t)^{*}$$

$$= \frac{1}{2}\int_{-\infty}^{+\infty}\frac{d\omega}{2\pi}\,e^{i\omega t}\,\vec{E}(-\omega)\cdot\frac{\partial}{\partial t}\int_{-\infty}^{+\infty}\frac{d\omega'}{2\pi}\,e^{-i\omega' t}\,\overleftrightarrow{\epsilon}(\omega')\cdot\vec{E}(\omega')$$

$$+ \frac{1}{2}\int_{-\infty}^{+\infty}\frac{d\omega}{2\pi}\,e^{-i\omega t}\,\vec{E}(\omega)\cdot\frac{\partial}{\partial t}\int_{-\infty}^{+\infty}\frac{d\omega'}{2\pi}\,e^{i\omega' t}\,\overleftrightarrow{\epsilon}(-\omega')\cdot\vec{E}(-\omega')$$

$$= \frac{1}{2}\int_{-\infty}^{+\infty}\frac{d\omega}{2\pi}\int_{-\infty}^{+\infty}\frac{d\omega'}{2\pi}\,e^{-i(\omega'-\omega)t}\,(-i\omega')\,\vec{E}(-\omega)\cdot\overleftrightarrow{\epsilon}(\omega')\cdot\vec{E}(\omega')$$

$$+ \frac{1}{2}\int_{-\infty}^{+\infty}\frac{d\omega}{2\pi}\int_{-\infty}^{+\infty}\frac{d\omega'}{2\pi}\,e^{-i(\omega-\omega')t}\,(i\omega')\,\vec{E}(\omega)\cdot\overleftrightarrow{\epsilon}(-\omega')\cdot\vec{E}(-\omega')$$

switching $\omega \leftrightarrow \omega'$ in second term

$$= \frac{1}{2}\int_{-\infty}^{+\infty}\frac{d\omega}{2\pi}\int_{-\infty}^{+\infty}\frac{d\omega'}{2\pi}\,e^{-i(\omega'-\omega)t}\,(-i\omega')\,\vec{E}(-\omega)\cdot\overleftrightarrow{\epsilon}(\omega')\cdot\vec{E}(\omega')$$

$$+ \frac{1}{2}\int_{-\infty}^{+\infty}\frac{d\omega}{2\pi}\int_{-\infty}^{+\infty}\frac{d\omega'}{2\pi}\,e^{-i(\omega'-\omega)t}\,(i\omega)\,\vec{E}(\omega')\cdot\overleftrightarrow{\epsilon}(-\omega)\cdot\vec{E}(-\omega)$$

using $E_i(\omega')\,\epsilon_{ij}(-\omega)\,E_j(-\omega) = E_j(-\omega)\,\epsilon_{ji}(-\omega)\,E_i(\omega')$

$$= \frac{1}{2}\int_{-\infty}^{+\infty}\frac{d\omega}{2\pi}\int_{-\infty}^{+\infty}\frac{d\omega'}{2\pi}\,e^{-i(\omega'-\omega)t}\,(-i\omega')\,\vec{E}(-\omega)\cdot\overleftrightarrow{\epsilon}(\omega')\cdot\vec{E}(\omega')$$

$$+ \frac{1}{2}\int_{-\infty}^{+\infty}\frac{d\omega}{2\pi}\int_{-\infty}^{+\infty}\frac{d\omega'}{2\pi}\,e^{-i(\omega'-\omega)t}\,(i\omega)\,\vec{E}(-\omega)\cdot\overleftrightarrow{\epsilon}^{T}(-\omega)\cdot\vec{E}(\omega')$$

$$= \frac{1}{2}\int_{-\infty}^{+\infty}\frac{d\omega}{2\pi}\int_{-\infty}^{+\infty}\frac{d\omega'}{2\pi}\,e^{-i(\omega'-\omega)t}\,\vec{E}(-\omega)\cdot(-i)\left[\omega'\,\overleftrightarrow{\epsilon}(\omega') - \omega\,\overleftrightarrow{\epsilon}(-\omega)\right]\cdot\vec{E}(\omega')$$

21 Assuming no absorption for simplicity we have.

$$\overleftrightarrow{\epsilon}(-\omega) = \overleftrightarrow{\epsilon}(\omega).$$

Thus,

$$\vec{E}(t)\cdot\frac{\partial}{\partial t}\vec{D}(t) = \frac{1}{2}\int_{-\infty}^{+\infty}\frac{d\omega}{2\pi}\int_{-\infty}^{+\infty}\frac{d\omega'}{2\pi}\, e^{-i(\omega'-\omega)t}\,\vec{E}(-\omega)\cdot(-i)\left[\omega'\overleftrightarrow{\epsilon}(\omega') - \omega\overleftrightarrow{\epsilon}(\omega)\right]\cdot\vec{E}(\omega')$$

$$= \frac{\partial}{\partial t}\frac{1}{2}\int_{-\infty}^{+\infty}\frac{d\omega}{2\pi}\int_{-\infty}^{+\infty}\frac{d\omega'}{2\pi}\, e^{-i(\omega'-\omega)t}\,\vec{E}(-\omega)\cdot\frac{\left[\omega'\overleftrightarrow{\epsilon}(\omega') - \omega\overleftrightarrow{\epsilon}(\omega)\right]}{(\omega'-\omega)}\cdot\vec{E}(\omega')$$

22 Similarly, we can show that

$$\vec{H}(t)\cdot\frac{\partial}{\partial t}\vec{B}(t) = \frac{\partial}{\partial t}\frac{1}{2}\int_{-\infty}^{+\infty}\frac{d\omega}{2\pi}\int_{-\infty}^{+\infty}\frac{d\omega'}{2\pi}\, e^{-i(\omega'-\omega)t}\,\vec{H}(-\omega)\cdot\frac{\left[\omega'\overleftrightarrow{\mu}(\omega') - \omega\overleftrightarrow{\mu}(\omega)\right]}{(\omega'-\omega)}\cdot\vec{H}(\omega')$$

23 Using 21 and 22 we identify

$$\vec{E}(t)\cdot\frac{\partial}{\partial t}\vec{D}(t) + \vec{H}(t)\cdot\frac{\partial}{\partial t}\vec{B}(t) = \frac{\partial}{\partial t}U(t)$$

where.

$$U(t) = \int_{-\infty}^{+\infty}\frac{d\omega}{2\pi}\int_{-\infty}^{+\infty}\frac{d\omega'}{2\pi}\, e^{-i(\omega'-\omega)t}\,U(\omega,\omega')$$

$$U(\omega,\omega') = \frac{1}{2}\vec{E}(-\omega)\cdot\frac{\left[\omega'\overleftrightarrow{\epsilon}(\omega') - \omega\overleftrightarrow{\epsilon}(\omega)\right]}{(\omega'-\omega)}\cdot\vec{E}(\omega')$$

$$+ \frac{1}{2}\vec{H}(-\omega)\cdot\frac{\left[\omega'\overleftrightarrow{\mu}(\omega') - \omega\overleftrightarrow{\mu}(\omega)\right]}{(\omega'-\omega)}\cdot\vec{H}(\omega')$$

(24) Note that for constant $\overleftrightarrow{\epsilon}(\omega) = \overleftrightarrow{\epsilon}$ and $\overleftrightarrow{\mu}(\omega) = \overleftrightarrow{\mu}$

we recover

$$U(\omega,\omega') = \frac{1}{2}\,\vec{E}(-\omega)\cdot\overleftrightarrow{\epsilon}\cdot\vec{E}(\omega') + \frac{1}{2}\,\vec{H}(-\omega)\cdot\overleftrightarrow{\mu}\cdot\vec{H}(\omega').$$

(25) Now consider a slowly evolving field, such that the range of frequencies that contribute to $\int d\omega \int d\omega'$ are very small. Then,

$$\omega' - \omega = \Delta\omega$$

(26)

$$\frac{\omega'\,\overleftrightarrow{\epsilon}(\omega') - \omega\,\overleftrightarrow{\epsilon}(\omega)}{(\omega'-\omega)} = \frac{1}{\omega'-\omega}\left[\omega'\left\{\overleftrightarrow{\epsilon}(\omega) + \Delta\omega\frac{d}{d\omega}\overleftrightarrow{\epsilon}(\omega)\right\} - \omega\,\overleftrightarrow{\epsilon}(\omega)\right] \quad \text{for } \Delta\omega \ll \omega$$

$$= \overleftrightarrow{\epsilon}(\omega) + \omega'\frac{\Delta\omega}{\omega'-\omega}\frac{d}{d\omega}\overleftrightarrow{\epsilon}(\omega)$$

$$= \overleftrightarrow{\epsilon}(\omega) + \omega'\frac{d}{d\omega}\overleftrightarrow{\epsilon}(\omega)$$

$$= \overleftrightarrow{\epsilon}(\omega) + \omega\frac{d}{d\omega}\overleftrightarrow{\epsilon}(\omega) \qquad (\omega' = \omega + \underbrace{\Delta\omega}_{\text{neglect.}})$$

$$= \frac{d}{d\omega}\left[\omega\,\overleftrightarrow{\epsilon}(\omega)\right]$$

(27) Thus, for a slowly evolving field we have

$$U(\omega,\omega') = \frac{1}{2}\vec{E}(-\omega)\cdot\left[\frac{d}{d\omega}\left\{\omega\overleftrightarrow{\epsilon}(\omega)\right\}\right]\cdot\vec{E}(\omega) + \frac{1}{2}\vec{H}(-\omega)\cdot\overleftrightarrow{\mu}\cdot\vec{H}(\omega)$$

assuming $\mu = $ constant.

(28) Let us now investigate the speed of centroid of energy in a dispersive medium with

$$\frac{\epsilon(\omega)}{\epsilon_0} = 1 - \frac{\omega_p^2}{\omega^2}.$$

Notice

$$\frac{d}{d\omega}\left[\omega\frac{\epsilon(\omega)}{\epsilon_0}\right] = \frac{\epsilon(\omega)}{\epsilon_0} + \omega\frac{d}{d\omega}\frac{\epsilon(\omega)}{\epsilon_0}$$

$$= 1 - \frac{\omega_p^2}{\omega^2} + \omega\frac{d}{d\omega}\left(1 - \frac{\omega_p^2}{\omega^2}\right)$$

$$= 1 - \frac{\omega_p^2}{\omega^2} - \omega(-2)\frac{\omega_p^2}{\omega^3}$$

$$= 1 + \frac{\omega_p^2}{\omega^2}$$

(29)

$$|\vec{V}_E| = \frac{\int d^3r\,|\vec{E}\times\vec{H}|}{\int d^3r\;U}$$

monochromatic wave implies $\epsilon E^2 = \mu H^2$

$$V_E = \frac{EH}{U}$$

$$= \frac{\sqrt{\frac{\epsilon}{\mu}}}{\frac{1}{2}\frac{d}{d\omega}[\omega\epsilon] + \frac{1}{2}\epsilon}$$

$\mu = $ const.

30 For $\mu = \mu_0$ we have.

$$V_E = \frac{2\sqrt{\frac{\epsilon}{\mu_0}}}{\epsilon + \frac{d}{d\omega}(\omega\epsilon)}$$

$$= \frac{1}{\sqrt{\mu_0 \epsilon_0}} \frac{2\sqrt{\frac{\epsilon}{\epsilon_0}}}{\left[\frac{\epsilon}{\epsilon_0} + \frac{d}{d\omega}\left(\omega \frac{\epsilon}{\epsilon_0}\right)\right]}$$

$$\frac{V_E}{c} = \frac{2\sqrt{1 - \frac{\omega_p^2}{\omega^2}}}{\left[1 - \frac{\omega_p^2}{\omega^2} + 1 + \frac{\omega_p^2}{\omega^2}\right]}$$

$$\frac{V_E}{c} = \sqrt{1 - \frac{\omega_p^2}{\omega^2}} < 1 .$$