## Kramers - Kronig relation

1 $$\vec{P}(t) = \int_{-\infty}^{+\infty} dt' \; \chi(t-t') \; \vec{E}(t')$$

where

$$\chi(t-t') = \theta(t-t') \; f(t-t')$$

with

$$\theta(t-t') = \begin{cases} 1 & \text{if} \quad t > t' \\ 0 & \text{if} \quad t < t' \end{cases}$$

2 $$\tilde{\chi}(\omega) = \int_{-\infty}^{+\infty} dt \; e^{i\omega t} \; \chi(t)$$

$$= \int_{-\infty}^{+\infty} dt \; e^{i\omega t} \; \theta(t) \; f(t)$$

$$= \int_{-\infty}^{+\infty} dt \; e^{i\omega t} \int_{-\infty}^{+\infty} \frac{d\omega'}{2\pi} \; e^{-i\omega' t} \; \tilde{f}(\omega') \; \theta(t)$$

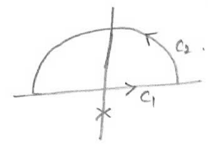
3 We shall now show that

$$\theta(t) = \underset{\delta \to 0+}{Lt} \; i \int_{-\infty}^{+\infty} \frac{d\omega}{2\pi} \; \frac{e^{-i\omega t}}{\omega + i\delta} = \begin{cases} 1 & \text{if} \quad t > 0 \\ 0 & \text{if} \quad t < 0 \end{cases}$$

(4) Consider the complex integral

$$i\int_C \frac{dz}{2\pi}\,\frac{e^{-izt}}{z+i\delta}$$

For $\underline{\underline{t<0}}$ consider the contour



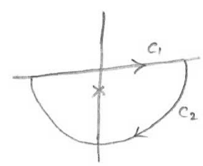

$$\int_{C_1} + \int_{C_2} = 0$$

$\sin\theta > 0$
$t<0$

$$\theta(t) + i\int_0^{\pi}\frac{d(Re^{i\theta})}{2\pi}\,\underbrace{\frac{e^{-itR\cos\theta}\,e^{tR\sin\theta}}{\left(Re^{i\theta}+i\delta\right)}}_{=0} = 0$$

$\theta(t) = 0$.

For $\underline{\underline{t>0}}$ consider the contour




$$\int_{C_1} + \int_{C_2} = 2\pi i\,\frac{i}{2\pi}\,e^{-it(-i\delta)}\,\underbrace{(-1)}_{\text{sign of contour}}$$

$\sin\theta < 0$
$t>0$

$$\theta(t) + i\int_0^{-\pi}\frac{d(Re^{i\theta})}{2\pi}\,\frac{e^{-itR\cos\theta}\,e^{tR\sin\theta}}{\left(Re^{i\theta}+i\delta\right)} = 1$$

$\theta(t) = 1$.

④ Using ③ in ②

$$\tilde{\chi}(\omega) = \int_{-\infty}^{+\infty} dt\, e^{i\omega t} \int_{-\infty}^{+\infty} \frac{d\omega'}{2\pi}\, e^{-i\omega' t}\, \tilde{f}(\omega') \underset{\delta\to 0+}{Lt}\; i \int_{-\infty}^{+\infty} \frac{d\omega''}{2\pi} \frac{e^{-i\omega'' t}}{\omega'' + i\delta}$$

$$= \underset{\delta\to 0+}{Lt}\; i \int_{-\infty}^{+\infty} \frac{d\omega'}{2\pi} \int_{-\infty}^{+\infty} \frac{d\omega''}{2\pi} \frac{\tilde{f}(\omega')}{\omega'' + i\delta} \int_{-\infty}^{+\infty} dt\, e^{it(\omega - \omega' - \omega'')}$$

$$= \underset{\delta\to 0+}{Lt}\; i \int_{-\infty}^{+\infty} \frac{d\omega'}{2\pi} \int_{-\infty}^{+\infty} \frac{d\omega''}{2\pi} \frac{\tilde{f}(\omega')}{\omega'' + i\delta}\, 2\pi\, \delta(\omega - \omega' - \omega'')$$

$$= \underset{\delta\to 0+}{Lt}\; i \int_{-\infty}^{+\infty} \frac{d\omega'}{2\pi} \frac{\tilde{f}(\omega')}{\omega - \omega' + i\delta}$$

⑤ We have the freedom to choose the form of $f(t)$ for $t < 0$. Let us choose

$$f(-t) = -f(t)$$

⑥

$$\tilde{f}(\omega) = \int_{-\infty}^{+\infty} dt\, e^{i\omega t} f(t)$$

$$= \int_{-\infty}^{0} dt\, e^{i\omega t} f(t) + \int_{0}^{\infty} dt\, e^{i\omega t} f(t)$$

$t = -t'$

$$= \int_{0}^{\infty} dt'\, e^{-i\omega t'} f(-t') + \int_{0}^{\infty} dt\, e^{i\omega t} f(t)$$

$$= -\int_{0}^{\infty} dt\, e^{-i\omega t} f(t) + \int_{0}^{\infty} dt\, e^{i\omega t} f(t)$$

$$= 2i \int_{0}^{\infty} dt\, (\sin \omega t)\, f(t)$$

(7) Using (6) we conclude that $\tilde{f}(\omega)$ is purely imaginary.

$$\tilde{f}(\omega) = i \ Im \ \tilde{f}(\omega).$$

or

$$Re \ \tilde{f}(\omega) = 0.$$

(8) Using (7) in (4) we have.

$$\tilde{\chi}(\omega) = \underset{\delta \to 0+}{Lt} \ i \int_{-\infty}^{+\infty} \frac{d\omega'}{2\pi} \frac{i \left[ Im \ \tilde{f}(\omega') \right]}{\omega - \omega' + i\delta}$$

$$= \underset{\delta \to 0+}{Lt} \int_{-\infty}^{+\infty} \frac{d\omega'}{2\pi} \frac{\left[ Im \ \tilde{f}(\omega') \right]}{\omega' - \omega - i\delta}$$

(9) Using (8) we have.

$$\left[ Im \ \tilde{\chi}(\omega) \right] = \int_{-\infty}^{+\infty} \frac{d\omega'}{2\pi} \left[ Im \ \tilde{f}(\omega') \right] \underset{\delta \to 0+}{Lt} \ Im \left( \frac{1}{\omega' - \omega - i\delta} \right)$$

(10) We shall next show that

$$\delta(x) = \frac{1}{\pi} \underset{\delta \to 0+}{Lt} \ Im \left( \frac{1}{x - i\delta} \right) \qquad \frac{x + i\delta}{x^2 + \delta^2}.$$

$$= \frac{1}{\pi} \underset{\delta \to 0+}{Lt} \frac{\delta}{x^2 + \delta^2}$$

(11) $\delta$-function needs to satisfy

(i) $\delta(x) = 0$ if $x \neq 0$.

(ii) $\delta(x) \to \infty$ if $x = 0$

(iii) $\int_{-\infty}^{+\infty} dx\, \delta(x) = 1$.

for $x \neq 0$, $\frac{1}{\pi} \underset{\delta \to 0+}{Lt} \frac{\delta}{x^2+\delta^2} = \frac{1}{\pi} \underset{\delta \to 0+}{Lt} \frac{\delta}{x^2} = 0$

for $x = 0$, $\frac{1}{\pi} \underset{\delta \to 0+}{Lt} \frac{\delta}{x^2+\delta^2} = \frac{1}{\pi} \underset{\delta \to 0+}{Lt} \frac{\delta}{\delta^2} \to \infty$.

$$\int_{-\infty}^{+\infty} dx\, \delta(x) = \frac{1}{\pi} \underset{\delta \to 0+}{Lt} \int_{-\infty}^{+\infty} dx \frac{\delta}{x^2+\delta^2}.$$

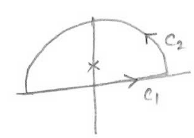
$x = \delta \tan\theta$.
$dx = \delta \sec^2\theta\, d\theta$.
$x^2 + \delta^2 = \delta^2 \sec^2\theta$.

$$= \frac{1}{\pi} \underset{\delta \to 0+}{Lt} \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \frac{\delta^2 \sec^2\theta\, d\theta}{\delta^2 \sec^2\theta}$$

$$= 1.$$

(12) Or, as a contour integral

$$\int_{-\infty}^{+\infty} dx\, \delta(x) = \frac{1}{\pi} \underset{\delta \to 0+}{Lt} \int_{-\infty}^{+\infty} dx\, Im\left(\frac{1}{x - i\delta}\right)$$




$$= \frac{1}{\pi} \underset{\delta \to 0+}{Lt}\, Im \int_{-\infty}^{+\infty} dx \frac{1}{x - i\delta}$$

$$= \frac{1}{\pi} \underset{\delta \to 0+}{Lt}\, Im \left[ - \int_0^{\pi} \frac{d(Re^{i\theta})}{Re^{i\theta} - i\delta} + 2\pi i \right]$$

$$= \frac{1}{\pi} \underset{\delta \to 0+}{Lt}\, Im \left[ -\pi i + 2\pi i \right]$$

$$= 1.$$

13 Thus, we have convinced ourselves that

$$\pi\,\delta(x) = \underset{\delta\to 0+}{Lt}\ \mathrm{Im}\left(\frac{1}{x-i\delta}\right)$$

14 Using 13 in 9

$$\left[\mathrm{Im}\ \tilde{\chi}(\omega)\right] = \int_{-\infty}^{+\infty} \frac{d\omega'}{2\pi}\left[\mathrm{Im}\ \tilde{f}(\omega')\right]\ \pi\ \delta(\omega'-\omega)$$

$$= \frac{1}{2}\left[\mathrm{Im}\ \tilde{f}(\omega)\right]$$

15 Using 14 in 8 we have.

$$\tilde{\chi}(\omega) = \underset{\delta\to 0+}{Lt}\int_{-\infty}^{+\infty}\frac{d\omega'}{2\pi}\ \frac{2\left[\mathrm{Im}\ \tilde{\chi}(\omega')\right]}{(\omega'-\omega-i\delta)}$$

16 $$\left[\mathrm{Re}\ \tilde{\chi}(\omega)\right] = \underset{\delta\to 0+}{Lt}\int_{-\infty}^{+\infty}\frac{d\omega'}{2\pi}\ 2\left[\mathrm{Im}\ \tilde{\chi}(\omega')\right]\ \mathrm{Re}\left(\frac{1}{\omega'-\omega-i\delta}\right)$$

$$= \underset{\delta\to 0+}{Lt}\int_{-\infty}^{+\infty}\frac{d\omega'}{2\pi}\ 2\left[\mathrm{Im}\ \tilde{\chi}(\omega')\right]\ \frac{(\omega'-\omega)}{(\omega'-\omega)^2+\delta^2}$$

which gives the relation between the real and imaginary part of the response. This is the Kramers-Kronig relation.