SET-06

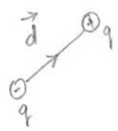
Macroscopic electrodynamics

1. Electric dipole: The simplest example is that of two equal and opposite charges separated by distance.

$$U_E = -\vec{d}\cdot\vec{E} \quad \rightarrow \text{energy}$$

$$\vec{F}_E = -\vec{\nabla}\, U_E = \vec{\nabla}(\vec{d}\cdot\vec{E}) \quad \rightarrow \text{force}$$

$$\vec{\tau}_E = \vec{d}\times\vec{E} \quad \rightarrow \text{torque}$$

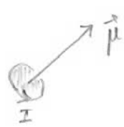


2. Magnetic dipole: The simplest example is a bar magnet (North pole and South pole). Another basis example is a circular current loop carrying current.

$$U_B = -\vec{\mu}\cdot\vec{B} \quad \rightarrow \text{energy}$$

$$\vec{F}_B = -\vec{\nabla}\, U_B = \vec{\nabla}(\vec{\mu}\cdot\vec{B}) \quad \rightarrow \text{force}$$

$$\vec{\tau}_B = \vec{\mu}\times\vec{B} \quad \rightarrow \text{torque}$$



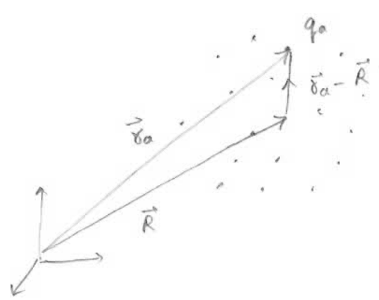
(3) Let us consider a neutral atom with charges e_a positioned at $\vec{r}_a$. Let $\vec{R}$ be the center-of-charge and $\vec{V}$ the velocity of center-of-charge. Let us define

$$\sum_a e_a = 0$$

$$\vec{d} = \sum_a e_a(\vec{r}_a - \vec{R}) = \sum_a e_a \vec{r}_a$$

$$\vec{\mu} = \frac{1}{2}\sum_a e_a (\vec{r}_a - \vec{R}) \times (\vec{V}_a - \vec{V})$$

(4)



The total Lorentz force on the atom in the presence of an electric and magnetic field is

$$\vec{F} = \sum_a \left[e_a \vec{E}(\vec{r}_a) + e_a \vec{V}_a \times \vec{B}(\vec{r}_a) \right]$$

5. Taylor expanding the fields around the center-of-charge we have

$$\vec{E}(\vec{r}_a) = \vec{E}(\vec{R}) + [(\vec{r}_a - \vec{R}) \cdot \vec{\nabla}]\, \vec{E}(\vec{R}) + \cdots$$

$$\vec{B}(\vec{r}_a) = \vec{B}(\vec{R}) + [(\vec{r}_a - \vec{R}) \cdot \vec{\nabla}]\, \vec{B}(\vec{R}) + \cdots$$

6. Approximations:

(i) $|(\vec{r}_a - \vec{R}) \cdot \vec{\nabla}| \rightarrow |(\vec{r}_a - \vec{R}) \cdot i\vec{k}_0| \ll 1$

where $\lambda_0 = \frac{2\pi}{|\vec{k}_0|}$ is a characteristic wavelength of the atom.

(ii) $|\vec{V}| \ll |\vec{v}_a| \ll c$.

7. Using 5 in 4

$$\vec{F} = \underbrace{\sum_a e_a}_{=0} \vec{E}(\vec{R}) + \overbrace{\sum_a e_a (\vec{r}_a - \vec{R})}^{\vec{d}} \cdot \vec{\nabla}\, \vec{E}(\vec{R})$$

$$+ \underbrace{\sum_a e_a \vec{v}_a}_{\frac{d}{dt}\vec{d}} \times \vec{B}(\vec{R}) + \sum_a e_a \vec{v}_a \times [(\vec{r}_a - \vec{R}) \cdot \vec{\nabla}]\, \vec{B}(\vec{R})$$

8 Thus, using 3 in 7

$$\vec{F} = \overbrace{(\vec{d}\cdot\vec{\nabla})\,\vec{E}(\vec{R})}^{2A} + \overbrace{\left(\frac{d}{dt}\vec{d}\right)\times\vec{B}(\vec{R})}^{1\,+\,2B\,+\,4A}$$

$$+ \sum_a e_a \vec{V}_a \times \underbrace{\left[(\vec{r}_a - \vec{R})\cdot\vec{\nabla}\right]\vec{B}(\vec{R})}_{3\,+\,4B}$$

9

$$\underbrace{\left(\frac{d}{dt}\vec{d}\right)\times\vec{B}(\vec{R})}_{1\,+\,2B\,+\,4A} = \frac{d}{dt}\left(\vec{d}\times\vec{B}(\vec{R})\right) - \vec{d}\times\left(\frac{d}{dt}\vec{B}(\vec{R})\right)$$

$$= \frac{d}{dt}\left(\vec{d}\times\vec{B}(\vec{R})\right) - \vec{d}\times\left[\left\{\frac{\partial}{\partial t} + \vec{V}\cdot\vec{\nabla}\right\}\vec{B}(\vec{R})\right]$$

$$= \frac{d}{dt}\left(\vec{d}\times\vec{B}(\vec{R})\right) - \vec{d}\times\underbrace{\left(\frac{\partial}{\partial t}\vec{B}(\vec{R})\right)}_{-\vec{\nabla}\times\vec{E}(\vec{R})} - \vec{d}\times[\vec{V}\cdot\vec{\nabla}]\,\vec{B}(\vec{R})$$

$$= \underbrace{\frac{d}{dt}\left(\vec{d}\times\vec{B}(\vec{R})\right)}_{1} + \underbrace{\vec{d}\times\left(\vec{\nabla}\times\vec{E}(\vec{R})\right)}_{2B} - \underbrace{\vec{d}\times[\vec{V}\cdot\vec{\nabla}]\,\vec{B}(\vec{R})}_{4A}$$

10 Using 9 in 8

$$\vec{F} = \underbrace{\frac{d}{dt}\left(\vec{d}\times\vec{B}(\vec{R})\right)}_{1} + \underbrace{(\vec{d}\cdot\vec{\nabla})\,\vec{E}(\vec{R})}_{2A} + \underbrace{\vec{d}\times\left(\vec{\nabla}\times\vec{E}(\vec{R})\right)}_{2B}$$

$$\underbrace{-\,\vec{d}\times[\vec{V}\cdot\vec{\nabla}]\,\vec{B}(\vec{R})}_{4A} + \underbrace{\sum_a e_a \vec{V}_a \times \left[(\vec{r}_a - \vec{R})\cdot\vec{\nabla}\right]\vec{B}(\vec{R})}_{3\,+\,4B}$$

11 We have

$$\underset{2\,B}{\vec{d} \times \left(\vec{\nabla} \times \vec{E}(\vec{R}) \right)} = \underset{2}{\vec{\nabla} \left(\vec{d} \cdot \vec{E}(\vec{R}) \right)} - \underset{2\,A}{(\vec{d} \cdot \vec{\nabla})\, \vec{E}(\vec{R})}$$

where we have used the fact that $\vec{d}$ is the property of the whole atom and independent of $\vec{R}$.

12 Using 11 in 10

$$\vec{F} = \underset{1}{\frac{d}{dt} \left(\vec{d} \times \vec{B}(\vec{R}) \right)} + \underset{2}{\vec{\nabla} \left[\vec{d} \cdot \vec{E}(\vec{R}) \right]}$$
$$- \underset{4\,A}{\vec{d} \times [\vec{V} \cdot \vec{\nabla}]\, \vec{B}(\vec{R})} + \underset{3\,+\,4\,B}{\sum_a e_a \vec{v}_a \times [(\vec{r}_a - \vec{R}) \cdot \vec{\nabla}]\, \vec{B}(\vec{R})}$$

13 Adding and subtracting a $\vec{V}$-dependent term we have.

$$\vec{F} = \underset{1}{\frac{d}{dt} \left(\vec{d} \times \vec{B}(\vec{R}) \right)} + \underset{2}{\vec{\nabla} \left[\vec{d} \cdot \vec{E}(\vec{R}) \right]}$$
$$- \underset{4\,A}{\vec{d} \times [\vec{V} \cdot \vec{\nabla}]\, \vec{B}(\vec{R})} + \underset{4\,B}{\sum_a e_a \vec{V} \times [(\vec{r}_a - \vec{R}) \cdot \vec{\nabla}]\, \vec{B}(\vec{R})}$$
$$+ \underset{3}{\sum_a e_a (\vec{v}_a - \vec{V}) \times [(\vec{r}_a - \vec{R}) \cdot \vec{\nabla}]\, \vec{B}(\vec{R})}$$

14 We process the 4A and 4B terms as.

$$-\vec{d} \times [\vec{V} \cdot \vec{\nabla}]\, \vec{B}(\vec{R}) + \vec{V} \times [\vec{d} \cdot \vec{\nabla}]\, \vec{B}(\vec{R})$$

$$\underset{4A}{} \qquad \underset{4B}{}$$

$$= \left[\vec{V}(\vec{d} \cdot \vec{\nabla}) - \vec{d}\,(\vec{V} \cdot \vec{\nabla}) \right] \times \vec{B}(\vec{R})$$

$$= \left[(\vec{d} \times \vec{V}) \times \vec{\nabla} \right] \times \vec{B}(\vec{R})$$

$$= \vec{\nabla}\left[(\vec{d} \times \vec{V}) \cdot \vec{B}(\vec{R}) \right] - (\vec{d} \times \vec{V})\, \underbrace{\vec{\nabla} \cdot \vec{B}(\vec{R})}_{=0}$$

$$\underset{4}{}$$

where we used $\vec{\nabla}\, \vec{d} = 0$ and $\vec{\nabla}\, \vec{V} = 0$ in the final step.

15 Using 14 in 13

$$\vec{F} = \underset{1}{\frac{d}{dt}\left(\vec{d} \times \vec{B}(\vec{R})\right)} + \underset{2}{\vec{\nabla}\left[\vec{d} \cdot \vec{E}(\vec{R}) \right]} + \underset{4}{\vec{\nabla}\left[(\vec{d} \times \vec{V}) \cdot \vec{B}(\vec{R}) \right]}$$

$$+ \underset{3}{\sum_a e_a (\vec{V}_a - \vec{V}) \times \left[(\vec{r}_a - \vec{R}) \cdot \vec{\nabla} \right] \vec{B}(\vec{R})}$$

16 We next process the term 3' as.

$$\underbrace{\sum_a e_a (\vec{V}_a - \vec{V}) \times [(\vec{r}_a - \vec{R}) \cdot \vec{\nabla}] \, \vec{B}(\vec{R})}_{3}$$

$$= \sum_a e_a \left(\frac{d}{dt} (\vec{r}_a - \vec{R}) \right) \times [(\vec{r}_a - \vec{R}) \cdot \vec{\nabla}] \, \vec{B}(\vec{R})$$

$$= \frac{d}{dt} \left[\sum_a e_a (\vec{r}_a - \vec{R}) \times [(\vec{r}_a - \vec{R}) \cdot \vec{\nabla}] \, \vec{B}(\vec{R}) \right] \longrightarrow \text{higher order of } 1.$$

$$- \sum_a e_a (\vec{r}_a - \vec{R}) \times [(\vec{V}_a - \vec{V}) \cdot \vec{\nabla}] \, \vec{B}(\vec{R})$$

$$- \sum_a e_a (\vec{r}_a - \vec{R}) \times [(\vec{r}_a - \vec{R}) \cdot \vec{\nabla}] \frac{d}{dt} \vec{B}(\vec{R}) \longrightarrow \text{higher order of } 2 \text{ and } 4.$$

17 Thus, we have.

$$\underbrace{\sum_a e_a (\vec{V}_a - \vec{V}) \times [(\vec{r}_a - \vec{R}) \cdot \vec{\nabla}] \, \vec{B}(\vec{R})}_{3} = \underbrace{- \sum_a e_a (\vec{r}_a - \vec{R}) \times [(\vec{V}_a - \vec{V}) \cdot \vec{\nabla}] \, \vec{B}(\vec{R})}_{3}$$

18 Using 17 we can write

$$\underbrace{\sum_a e_a (\vec{V}_a - \vec{V}) \times [(\vec{r}_a - \vec{R}) \cdot \vec{\nabla}] \, \vec{B}(\vec{R})}_{3}$$

$$= \frac{1}{2} \sum_a e_a (\vec{V}_a - \vec{V}) [(\vec{r}_a - \vec{R}) \cdot \vec{\nabla}] \times \vec{B}(\vec{R}) - \frac{1}{2} \sum e_a (\vec{r}_a - \vec{R}) [(\vec{V}_a - \vec{V}) \cdot \vec{\nabla}] \times \vec{B}(\vec{R})$$

$$= \frac{1}{2} \sum_a e_a \left[(\vec{V}_a - \vec{V}) \{(\vec{r}_a - \vec{R}) \cdot \vec{\nabla}\} - (\vec{r}_a - \vec{R}) \{(\vec{V}_a - \vec{V}) \cdot \vec{\nabla}\} \right] \times \vec{B}(\vec{R})$$

$$= \frac{1}{2} \sum_a e_a \left[\{(\vec{r}_a - \vec{R}) \times (\vec{V}_a - \vec{V})\} \times \vec{\nabla} \right] \times \vec{B}(\vec{R})$$

$$= \frac{1}{2} \sum_a e_a \left[\{(\vec{r}_a - \vec{R}) \times (\vec{V}_a - \vec{V})\} \times \vec{\nabla} \right]$$

19 Using 3 in 18

$$\sum_a e_a (\vec{v}_a - \vec{V}) \times \left[(\vec{r}_a - \vec{R}) \cdot \vec{\nabla} \right] \vec{B}(\vec{R}) = (\vec{\mu} \times \vec{\nabla}) \times \vec{B}(\vec{R})$$

3

$$= \vec{\nabla} \left[\vec{\mu} \cdot \vec{B}(\vec{R}) \right] - \vec{\mu} \; \underbrace{\vec{\nabla} \cdot \vec{B}(\vec{R})}_{=0}$$

3

20 Using 19 in 15 we have

$$\vec{F} = \frac{d}{dt} \left(\vec{d} \times \vec{B}(\vec{R}) \right)$$

1 → redefines momentum.

$$+ \vec{\nabla} \left[\vec{d} \cdot \vec{E} + \left\{ \vec{\mu} + \vec{d} \times \vec{V} \right\} \cdot \vec{B} \right]$$

2 → electric dipole

3 → magnetic dipole

4 → moving electric dipole acts as a magnetic dipole.

21 For $|\vec{V}| << |\vec{v}_a| << c$ we have

$$\vec{\mu} + \vec{d} \times \vec{V} \approx \vec{\mu}$$

Thus,

$$\vec{F} = \frac{d}{dt} (\vec{d} \times \vec{B}) + \vec{\nabla} \left[\vec{d} \cdot \vec{E} + \vec{\mu} \cdot \vec{B} \right].$$

Torque on an atom

(22)
$$\vec{\tau} = \sum_a (\vec{r}_a - \vec{R}) \times \vec{F}(\vec{r}_a)$$
$$= \sum_a (\vec{r}_a - \vec{R}) \times \left[e_a \vec{E}(\vec{r}_a) + e_a \vec{v}_a \times \vec{B}(\vec{r}_a) \right]$$
$$= \sum_a e_a (\vec{r}_a - \vec{R}) \times \vec{E}(\vec{R}) + \sum_a e_a (\vec{r}_a - \vec{R}) \times \left[\vec{v}_a \times \vec{B}(\vec{R}) \right]$$

where we neglected higher order in $(\vec{r}_a - \vec{R})$.

(23) Using (3) we have

$$\vec{\tau} = \vec{d} \times \vec{E}(\vec{R}) + \sum_a e_a (\vec{r}_a - \vec{R}) \times \left[\vec{v}_a \times \vec{B}(\vec{R}) \right]$$

(24) Processing the second term (using $|\vec{V}| << |\vec{v}_a|$)

$$\vec{\tau}_B = \sum_a e_a (\vec{r}_a - \vec{R}) \times \left[\vec{v}_a \times \vec{B}(\vec{R}) \right] = \sum_a e_a (\vec{r}_a - \vec{R}) \times \left[(\vec{v}_a - \vec{V}) \times \vec{B}(\vec{R}) \right]$$
$$= \frac{d}{dt}\left[\sum_a e_a (\vec{r}_a - \vec{R}) \times \left[(\vec{r}_a - \vec{R}) \times \vec{B}(\vec{R}) \right] \right]$$
$$- \sum_a e_a (\vec{v}_a - \vec{V}) \times \left[(\vec{r}_a - \vec{R}) \times \vec{B}(\vec{R}) \right]$$
$$- \sum_a e_a (\vec{v}_a - \vec{V}) \times \left[(\vec{r}_a - \vec{R}) \times \left(\frac{d}{dt} \vec{B}(\vec{R}) \right) \right]$$

higher order.

25 Let us write

$$\vec{\tau}_B = \sum_a e_a (\vec{r}_a - \vec{R}) \times [\vec{V}_a \times \vec{B}(\vec{R})]$$

$$= \frac{1}{2} \sum_a e_a (\vec{r}_a - \vec{R}) \times [(\vec{V}_a - \vec{V}) \times \vec{B}(\vec{R})] + \frac{1}{2} \sum_a e_a (\vec{r}_a - \vec{R}) \times [(\vec{V}_a - \vec{V}) \times \vec{B}(\vec{R})]$$

$$= \frac{1}{2} \sum_a e_a (\vec{r}_a - \vec{R}) \times [(\vec{V}_a - \vec{V}) \times \vec{B}(\vec{R})] - \frac{1}{2} \sum_a e_a (\vec{V}_a - \vec{V}) \times [(\vec{r}_a - \vec{R}) \times \vec{B}(\vec{R})]$$

$$+ \frac{1}{2} \frac{d}{dt} \left[\sum_a e_a (\vec{r}_a - \vec{R}) \times \{(\vec{r}_a - \vec{R}) \times \vec{B}(\vec{R})\} \right]$$

26 Using

$$\vec{A} \times (\vec{B} \times \vec{C}) + \vec{B} \times (\vec{C} \times \vec{A}) + \vec{C} \times (\vec{A} \times \vec{B}) = 0$$

we have

$$\vec{\tau}_B = \frac{1}{2} \sum_a e_a [(\vec{r}_a - \vec{R}) \times (\vec{V}_a - \vec{V})] \times \vec{B}(\vec{R}) + \frac{d}{dt} \sum_a \frac{1}{2} e_a (\vec{r}_a - \vec{R}) \times [(\vec{r}_a - \vec{R}) \times \vec{B}(\vec{R})]$$

$$= \vec{\mu} \times \vec{B} + \frac{d}{dt} \sum_a \frac{1}{2} e_a (\vec{r}_a - \vec{R}) \times [(\vec{r}_a - \vec{R}) \times \vec{B}(\vec{R})]$$

27 Using 26 in 23 we have

$$\vec{\tau} = \vec{d} \times \vec{E} + \vec{\mu} \times \vec{B} + \frac{d}{dt} \sum_a e_a (\vec{r}_a - \vec{R}) \times \left[\frac{1}{2} (\vec{r}_a - \vec{R}) \times \vec{B}(\vec{R}) \right]$$

$$\frac{d}{dt} \sum_a (\vec{r}_a - \vec{R}) \times m_a \vec{V}_a = \vec{d} \times \vec{E} + \vec{\mu} \times \vec{B} - \frac{d}{dt} \sum_a e_a (\vec{r}_a - \vec{R}) \times \left[\frac{1}{2} \vec{B}(\vec{R}) \times (\vec{r}_a - \vec{R}) \right]$$

$$\frac{d}{dt} \sum_a (\vec{r}_a - \vec{R}) \times \left[m_a \vec{V}_a + e_a \frac{1}{2} \vec{B}(\vec{R}) \times (\vec{r}_a - \vec{R}) \right] = \vec{d} \times \vec{E} + \vec{\mu} \times \vec{B}$$

(28) With respect to the origin at $\vec{R}$

$$\vec{A}(\vec{r}_a - \vec{R}) = \frac{1}{2}\,\vec{B}(\vec{R}) \times (\vec{r}_a - \vec{R})$$

which satisfies

$$\vec{B} = \vec{\nabla} \times \vec{A}$$

for constant magnetic fields. Thus, we can write

$$\frac{d}{dt} \sum_a (\vec{r}_a - \vec{R}) \times \left[m_a \vec{v}_a + e_a \vec{A}(\vec{r}_a - \vec{R}) \right] = \vec{d} \times \vec{E} + \vec{\mu} \times \vec{B}$$

which involves the canonical momentum

$$\vec{P} = m\vec{v} + e\vec{A}.$$

Force on a macroscopic body

29 We have calculated the force on an atom to be

$$\vec{F}_{atom} = \frac{d}{dt}(\vec{d}\times\vec{B}) + \vec{\nabla}\left[\vec{d}\cdot\vec{E} + \vec{\mu}\cdot\vec{B}\right]$$

30 The total force on a macroscopic body will be

$$\vec{F} = \int d^3r \; n(\vec{r}) \; \vec{F}_{atom}(\vec{r})$$

where $n(\vec{r})$ is the number of atoms per unit volume.

31 We used $\vec{\nabla}\,\vec{d} = 0$, but for macroscopic body it can vary with position. Thus, we unwind those steps by replacing

$$\vec{\nabla}(\vec{d}\cdot\vec{E}) = \vec{d}\times(\vec{\nabla}\times\vec{E}) + (\vec{d}\cdot\vec{\nabla})\,\vec{E}$$

$$\vec{\nabla}(\vec{\mu}\cdot\vec{B}) = \vec{\mu}\times(\vec{\nabla}\times\vec{B}) + (\vec{\mu}\cdot\vec{\nabla})\,\vec{B}$$

32 Using 29, 30 and 31

$$\vec{F} = \int d^3r \; n(\vec{r}) \left[\frac{d}{dt}(\vec{d}\times\vec{B}) + \vec{d}\times(\vec{\nabla}\times\vec{E}) + (\vec{d}\cdot\vec{\nabla})\,\vec{E} + \vec{\mu}\times(\vec{\nabla}\times\vec{B}) + (\vec{\mu}\cdot\vec{\nabla})\,\vec{B} \right]$$

(33) Define:

$$\vec{P}(\vec{r},t) = n(\vec{r})\ \vec{d}(\vec{r},t)$$

$$\vec{M}(\vec{r},t) = n(\vec{r})\ \vec{\mu}(\vec{r},t)$$

(34) in terms of which we have

$$\vec{F} = \int d^3r \left[\frac{\partial}{\partial t}(\vec{P}\times\vec{B}) + \vec{P}\times(\vec{\nabla}\times\vec{E}) + (\vec{P}\cdot\vec{\nabla})\vec{E} + \vec{M}\times(\vec{\nabla}\times\vec{B}) + (\vec{M}\cdot\vec{\nabla})\vec{B} \right]$$

(35) we used

$$\frac{d}{dt} \sim \frac{\partial}{\partial t}$$

which is true for $|\vec{V}| << |\vec{V}_a| << c$.

(36)
$$\frac{\partial}{\partial t}(\vec{P}\times\vec{B}) + \vec{P}\times(\vec{\nabla}\times\vec{E}) = \frac{\partial}{\partial t}(\vec{P}\times\vec{B}) - \vec{P}\times\frac{\partial\vec{B}}{\partial t}$$
$$= \left(\frac{\partial\vec{P}}{\partial t}\right)\times\vec{B}$$

using $\vec{\nabla}\times\vec{E} = -\frac{\partial\vec{B}}{\partial t}$

(37)
$$\vec{M}\times(\vec{\nabla}\times\vec{B}) + \vec{B}\times(\vec{\nabla}\times\vec{M})$$
$$= \vec{M}\,\vec{\nabla}\,\vec{B} - (\vec{M}\cdot\vec{\nabla})\vec{B} + \vec{B}\,\vec{\nabla}\,\vec{M} - (\vec{B}\cdot\vec{\nabla})\vec{M}$$
$$= \vec{\nabla}(\vec{M}\cdot\vec{B}) - (\vec{M}\cdot\vec{\nabla})\vec{B} - (\vec{B}\cdot\vec{\nabla})\vec{M}$$

38 Using 36 and 37 in 34

$$\vec{F} = \int d^3r \left[\left(\frac{\partial \vec{P}}{\partial t}\right) \times \vec{B} + (\vec{P}\cdot\vec{\nabla})\vec{E} + \vec{\nabla}(\vec{M}\cdot\vec{B}) - (\vec{B}\cdot\vec{\nabla})\vec{M} + (\vec{\nabla}\times\vec{M})\times\vec{B} \right]$$

39 On the boundary of the macroscopic body we have $n(\vec{r}) = 0$, thus $\vec{P}$ and $\vec{M}$ go to zero at these boundaries.

40 Using this we have.

$$\int d^3r \, \vec{\nabla}(\vec{M}\cdot\vec{B}) = 0$$

$$\int d^3r \, (\vec{P}\cdot\vec{\nabla})\vec{E} = -\int d^3r \, (\vec{\nabla}\cdot\vec{P})\vec{E} + \underbrace{\int d^3r \, \vec{\nabla}\cdot(\vec{P}\,\vec{E})}_{=0}$$

$$= -\int d^3r \, (\vec{\nabla}\cdot\vec{P})\vec{E}$$

$$\int d^3r \, (\vec{B}\cdot\vec{\nabla})\vec{M} = -\int d^3r \, \underbrace{(\vec{\nabla}\cdot\vec{B})}_{=0}\vec{M} + \underbrace{\int d^3r \, \vec{\nabla}\cdot(\vec{B}\,\vec{M})}_{=0}$$

$$= 0$$

41 Using 40 in 38

$$\vec{F} = \int d^3r \left[\left(\frac{\partial \vec{P}}{\partial t}\right) \times \vec{B} - (\vec{\nabla}\cdot\vec{P})\vec{E} + (\vec{\nabla}\times\vec{M})\times\vec{B} \right]$$

$$= \int d^3r \left[-(\vec{\nabla}\cdot\vec{P})\vec{E} + \left\{ \frac{\partial \vec{P}}{\partial t} + \vec{\nabla}\times\vec{M} \right\} \times \vec{B} \right]$$

42 Comparing 41 with the expression for the Lorentz force on a continuous distribution of charges,

$$\vec{F} = \int d^3r \left[\rho \vec{E} + \vec{J} \times \vec{B} \right]$$

we identify the effective charge densities and current densities for a macroscopic body as

$$\rho_{eff}(\vec{r},t) = - \vec{\nabla} \cdot \vec{P}(\vec{r},t)$$

$$\vec{J}_{eff}(\vec{r},t) = \frac{\partial}{\partial t} \vec{P}(\vec{r},t) + \vec{\nabla} \times \vec{M}(\vec{r},t) .$$

43 Thus, the macroscopic Maxwell's equations will be

$$\vec{\nabla} \cdot \epsilon_0 \vec{E} = \rho - \vec{\nabla} \cdot \vec{P}$$

$$\vec{\nabla} \cdot \vec{B} = 0$$

$$\vec{\nabla} \times \vec{E} = - \frac{\partial \vec{B}}{\partial t}$$

$$\vec{\nabla} \times \frac{1}{\mu_0} \vec{B} = \frac{\partial}{\partial t} (\epsilon_0 \vec{E}) + \vec{J} + \frac{\partial}{\partial t} \vec{P} + \vec{\nabla} \times \vec{M}$$

(44) Define

$$\vec{D} = \epsilon_0 \vec{E} + \vec{P}$$

$$\vec{H} = \frac{1}{\mu_0} \vec{B} - \vec{M}$$

(45) which leads to

$$\vec{\nabla} \cdot \vec{D} = \rho$$

$$\vec{\nabla} \cdot \vec{B} = 0$$

$$\vec{\nabla} \times \vec{E} = -\frac{\partial \vec{B}}{\partial t}$$

$$\vec{\nabla} \times \vec{H} = \frac{\partial \vec{D}}{\partial t} + \vec{J} .$$