SET-05

Conservation laws

(1) Maxwell's equations in SI units are.

$$\vec{\nabla}\cdot\vec{D} = \rho_e \qquad \vec{\nabla}\cdot\vec{B} = 0 \qquad \vec{D} = \epsilon_0 \vec{E}$$

$$-\vec{\nabla}\times\vec{E} - \frac{\partial \vec{B}}{\partial t} = 0 \qquad \vec{\nabla}\times\vec{H} - \frac{\partial \vec{D}}{\partial t} = \vec{J}_e \qquad \vec{H} = \frac{1}{\mu_0}\vec{B}$$

and

$$\vec{F} = q_e\left[\vec{E} + \vec{v}\times\vec{B}\right]$$

(2) It will be convenient to introduce magnetic monopoles for our discussion. In the presence of magnetic monopoles the Maxwell's equations generalize to

$$\vec{\nabla}\cdot\vec{D} = \rho_e \qquad \vec{\nabla}\cdot\vec{B} = \rho_m \qquad \vec{D} = \epsilon_0 \vec{E}$$

$$-\vec{\nabla}\times\vec{E} - \frac{\partial \vec{B}}{\partial t} = \vec{J}_m \qquad \vec{\nabla}\times\vec{H} - \frac{\partial \vec{D}}{\partial t} = \vec{J}_e \qquad \vec{H} = \frac{1}{\mu_0}\vec{B}$$

and

$$\vec{F} = q_e\left[\vec{E} + \vec{v}\times\vec{B}\right] + q_m\left[\vec{H} - \vec{v}\times\vec{D}\right],$$

(3) which is suggested by the invariance under

$$\begin{pmatrix}\rho_e \\ \vec{J}_e \\ \vec{E}\end{pmatrix} \to \begin{pmatrix}\rho_m \\ \vec{J}_m \\ \vec{B}\end{pmatrix} \quad \text{and} \quad \begin{pmatrix}\rho_m \\ \vec{J}_m \\ \vec{B}\end{pmatrix} \to \begin{pmatrix}-\rho_e \\ -\vec{J}_e \\ -\vec{E}\end{pmatrix}.$$

(4) The correspondence with Gaussian and Lorentz-Heaviside units is obtained by

$$(q_m)_G = \frac{1}{\sqrt{4\pi\mu_0}} (q_m)_{SI} = \frac{1}{\sqrt{4\pi}} (q_m)_{LH}$$

$$(\vec{J}_m)_G = \frac{1}{\sqrt{4\pi\mu_0}} (\vec{J}_m)_{SI} = \frac{1}{\sqrt{4\pi}} (\vec{J}_m)_{LH}$$

(5) $$\text{Power} = \frac{\text{Energy}}{\text{time}} = \frac{F\,d}{t} \longrightarrow \vec{F}\cdot\vec{V}$$

Thus, the rate at which work is done a particle is

$$\vec{F}\cdot\vec{V} = q_e\left[\vec{E} + \vec{V}\times\vec{B}\right]\cdot\vec{V} + q_m\left[\vec{H} - \vec{V}\times\vec{D}\right]\cdot\vec{V}$$

$$= q_e\,\vec{E}\cdot\vec{V} + q_m\,\vec{H}\cdot\vec{V}$$

(6) The charge density and current density for a single particle is given by.

$$\rho_e(\vec{x}) = q_e\,\delta^{(3)}(\vec{x} - \vec{x}_a(t))$$

$$\rho_m(\vec{x}) = q_m\,\delta^{(3)}(\vec{x} - \vec{x}_a(t))$$

$$\vec{J}_e(t) = q_e\,\vec{V}_a(t)\,\delta^{(3)}(\vec{x} - \vec{x}_a(t))$$

$$\vec{J}_m(t) = q_m\,\vec{V}_a(t)\,\delta^{(3)}(\vec{x} - \vec{x}_a(t))$$

I Conservation of electromagnetic energy

7. Using 6 in 5 we can write

$$\vec{F}\cdot\vec{v} = \int d^3x \left[\vec{j}_e \cdot \vec{E} + \vec{j}_m \cdot \vec{H} \right]$$

8. Using 2 in 7 to replace $\vec{j}_e$ and $\vec{j}_m$

$$\vec{j}_e\cdot\vec{E} + \vec{j}_m\cdot\vec{H} = \left[\vec{\nabla}\times\vec{H} - \frac{\partial\vec{D}}{\partial t}\right]\cdot\vec{E} + \left[-\vec{\nabla}\times\vec{E} - \frac{\partial\vec{B}}{\partial t}\right]\cdot\vec{H}$$

$$= -\frac{\partial}{\partial t}\left(\frac{1}{2}\epsilon_0 E^2 + \frac{1}{2}\mu_0 H^2\right) + (\vec{\nabla}\times\vec{H})\cdot\vec{E} - (\vec{\nabla}\times\vec{E})\cdot\vec{H}$$

9.
$$\vec{\nabla}\cdot(\vec{E}\times\vec{H}) = \epsilon_{ijk}\,\nabla_i (E_j H_k)$$

$$= \epsilon_{ijk}(\nabla_i E_j) H_k + \epsilon_{ijk} E_j (\nabla_i H_k)$$

$$= (\vec{\nabla}\times\vec{E})\cdot\vec{H} - \vec{E}\cdot(\vec{\nabla}\times\vec{H})$$

10. Using 9 in 8 we have.

$$\vec{j}_e\cdot\vec{E} + \vec{j}_m\cdot\vec{H} = -\frac{\partial}{\partial t}\left(\frac{1}{2}\epsilon_0 E^2 + \frac{1}{2}\mu_0 H^2\right) - \vec{\nabla}\cdot(\vec{E}\times\vec{H})$$

11. Define.

$U = \frac{1}{2}\epsilon_0 E^2 + \frac{1}{2}\mu_0 H^2 \longrightarrow$ electromagnetic energy density

$\vec{S} = \vec{E}\times\vec{H} \longrightarrow$ electromagnetic energy flux vector (Poynting vector).

(12)(a) Thus we have.

$$\frac{\partial}{\partial t} U + \vec{\nabla} \cdot \vec{S} + \underbrace{\vec{j}_e \cdot \vec{E} + \vec{j}_m \cdot \vec{H}}_{} = 0$$

rate of change of EM energy inside volume V.

rate of flow of EM energy out of volume V.

rate of transfer of EM energy t charge particles inside volume V.

This is the statement of conservation of electromagnetic energy.

(12)(b) The above interpretation comes out by integrating over a volume V.

$$\int_V d^3x \, \frac{\partial}{\partial t} U + \int_V d^3x \, \vec{\nabla} \cdot \vec{S} + \int_V d^3x \left(\vec{j}_e \cdot \vec{E} + \vec{j}_m \cdot \vec{H} \right) = 0$$

$$\frac{\partial}{\partial t} \left(\int_V d^3x \, U \right) + \oint_S d\vec{a} \cdot \vec{S} + \int_V d^3x \left(\vec{j}_e \cdot \vec{E} + \vec{j}_m \cdot \vec{H} \right) = 0$$

II Conservation of momentum

(13) Force on a particle is

$$\vec{F} = q_e\left[\vec{E} + \vec{v}\times\vec{B}\right] + q_m\left[\vec{H} - \vec{v}\times\vec{D}\right]$$

$$= \int d^3x \left[\, \rho_e \vec{E} + \rho_m \vec{H} + \vec{j}_e\times\vec{B} - \vec{j}_m \times \vec{D}\right]$$

(14) We can write

$$\vec{F} = \int d^3x \, \vec{f}$$

where $\vec{f}$ is the force density,

$$\vec{f} = \rho_e \vec{E} + \rho_m \vec{H} + \vec{j}_e \times \vec{B} - \vec{j}_m \times \vec{D}$$

(15) Using Maxwell's equations in (2) we have.

$$\vec{f} = (\vec{\nabla}\cdot\vec{D})\,\vec{E} + (\vec{\nabla}\cdot\vec{B})\,\vec{H} + \left[(\vec{\nabla}\times\vec{H}) - \frac{\partial\vec{D}}{\partial t}\right]\times\vec{B} - \left[-(\vec{\nabla}\times\vec{E}) - \frac{\partial\vec{B}}{\partial t}\right]\times\vec{D}$$

$$= -\frac{\partial}{\partial t}\left(\vec{D}\times\vec{B}\right) + (\vec{\nabla}\cdot\vec{D})\,\vec{E} + (\vec{\nabla}\times\vec{E})\times\vec{D} + (\vec{\nabla}\cdot\vec{B})\,\vec{H} + (\vec{\nabla}\times\vec{H})\times\vec{B}$$

(16)
$$\vec{f} = -\frac{\partial}{\partial t}\left(\vec{D}\times\vec{B}\right)$$
$$+\ \vec{E}\,(\vec{\nabla}\cdot\vec{D}) - \vec{D}\times(\vec{\nabla}\times\vec{E})$$
$$+\ \vec{H}\,(\vec{\nabla}\cdot\vec{B}) - \vec{B}\times(\vec{\nabla}\times\vec{H})$$

(17)
$$\vec{E}\,(\vec{\nabla}\cdot\vec{D}) - \vec{D}\times(\vec{\nabla}\times\vec{E})$$
$$= E_i\,\nabla_j D_j - \epsilon_{ijk}\, D_j\, \epsilon_{kmn}\, \nabla_m E_n.$$
$$= E_i\,\nabla_j D_j - \left(\delta_{im}\delta_{jn} - \delta_{in}\delta_{jm}\right) D_j\, \nabla_m E_n.$$
$$= E_i\,\nabla_j D_j - D_j\,\nabla_i E_j + D_j\,\nabla_j E_i$$
$$= -\frac{1}{2}\nabla_i\left(\epsilon_0 E^2\right) + E_i\,\nabla_j D_j + D_j\,\nabla_j E_i$$
$$= -\vec{\nabla}\left(\frac{1}{2}\epsilon_0 E^2\right) + \vec{\nabla}\cdot\left(\vec{D}\;\vec{E}\right)$$

free index.

(18) Similarly
$$\vec{H}\,(\vec{\nabla}\cdot\vec{B}) - \vec{B}\times(\vec{\nabla}\times\vec{H}) = -\vec{\nabla}\left(\frac{1}{2}\mu_0 H^2\right) - \vec{\nabla}\cdot\left(\vec{B}\;\vec{H}\right)$$

(19) Using (17) and (18) in (16)
$$\vec{f} = -\frac{\partial}{\partial t}\left(\vec{D}\times\vec{B}\right)$$
$$-\ \vec{\nabla}\left(\frac{1}{2}\epsilon_0 E^2 + \frac{1}{2}\epsilon_0 H^2\right)$$
$$+\ \vec{\nabla}\cdot\left(\vec{D}\;\vec{E} + \vec{B}\;\vec{H}\right)$$

20 Define:

$$\vec{G} = \vec{D} \times \vec{B}.$$
$$= \epsilon_0 \mu_0 \, \vec{E} \times \vec{H}$$
$$= \frac{1}{c^2} \vec{S} \longrightarrow \text{electromagnetic momentum density}$$

$$\overleftrightarrow{T} = \overleftrightarrow{1} \left(\frac{1}{2} \vec{D} \cdot \vec{E} + \frac{1}{2} \vec{B} \cdot \vec{H} \right) - \left(\vec{D} \, \vec{E} + \vec{B} \, \vec{H} \right)$$

$\searrow$ electromagnetic momentum flux tensor (stress tensor)

21 In terms of 20 we can write 21 as

$$\vec{f} + \frac{\partial}{\partial t} \vec{G} + \vec{\nabla} \cdot \overleftrightarrow{T} = 0$$

22
$$\int_V d^3x \, \vec{f} + \frac{\partial}{\partial t} \int_V d^3x \, \vec{G} + \int_S d\vec{a} \cdot \overleftrightarrow{T} = 0$$

force on the particles inside volume V.

rate of change of EM momentum inside volume V.

stress on the boundary of volume V.

23 Couple of important properties of stress tensor,

$$\overleftrightarrow{T} = \overleftrightarrow{I}\left(\frac{1}{2}\vec{D}\cdot\vec{E} + \frac{1}{2}\vec{B}\cdot\vec{H}\right) - \left(\vec{D}\,\vec{E} + \vec{B}\,\vec{H}\right)$$

Remember, we are considering the case

$\vec{D} = \epsilon_0 \vec{E}$ and $\vec{B} = \mu_0 \vec{H}$.

(i) $\overleftrightarrow{T}$ is symmetric, $T_{ij} = T_{ji}$.

(ii)
$$\begin{aligned} Tr(\overleftrightarrow{T}) &= T_{ii} \\ &= 3\left(\frac{1}{2}\vec{D}\cdot\vec{E} + \frac{1}{2}\vec{B}\cdot\vec{H}\right) - \left(\vec{D}\cdot\vec{E} + \vec{B}\cdot\vec{H}\right) \\ &= \frac{1}{2}\left(\vec{D}\cdot\vec{E} + \vec{B}\cdot\vec{H}\right) \\ &= U. \end{aligned}$$

24 $\vec{S} = \vec{E}\times\vec{H} \rightarrow$ energy $\times$ velocity $\rightarrow$ energy flux vector $\rightarrow \frac{\partial U}{\partial t} + \vec{\nabla}\cdot\vec{S}$.

$\vec{G} = \vec{D}\times\vec{B} \rightarrow$ mass $\times$ velocity $\rightarrow$ momentum density $\rightarrow \vec{f} + \frac{\partial \vec{G}}{\partial t}$

$$\vec{G} = \epsilon_0\mu_0 \vec{S} = \frac{1}{c^2}\vec{S}$$

Dimensional analysis suggests

$$(\text{mass}) \times c^2 = \text{energy}.$$

This is the first indication of relativistic connection between energy and mass, $E = mc^2$.

III Conservation of angular momentum

(25) Torque on a particle is

$$\vec{\tau} = \vec{r}_a \times \vec{F}(\vec{r}_a)$$
$$= \int d^3r \; \vec{r} \times \vec{f}$$

(26) We have seen

$$\frac{\partial}{\partial t} \vec{G} + \vec{\nabla} \cdot \overleftrightarrow{T} + \vec{f} = 0$$

(27)

$$\frac{\partial}{\partial t} G_k + \nabla_m T_{mk} + f_k = 0$$

$$x_j \frac{\partial}{\partial t} G_k + x_j \nabla_m T_{mk} + x_j f_k = 0$$

$$\frac{\partial}{\partial t} (x_j G_k) + \nabla_m (x_j T_{mk}) - (\nabla_m x_j) T_{mk} + x_j f_k = 0$$

$$\frac{\partial}{\partial t} (x_j G_k) + \nabla_m (T_{mk} x_j) - \delta_{mj} T_{mk} + x_j f_k = 0$$

$$\frac{\partial}{\partial t} (x_j G_k) + \nabla_m (T_{mk} x_j) - T_{jk} + x_j f_k = 0$$

28 Multiplying by ϵ_{ijk} and noting that

$$\epsilon_{ijk} T_{jk} = 0 \qquad \text{(since } T_{jk} \text{ is symmetric)}$$

we have.

$$\frac{\partial}{\partial t}(\vec{x} \times \vec{G}) + \vec{\nabla} \cdot (-\overleftrightarrow{T} \times \vec{x}) + \underbrace{\vec{x} \times \vec{f}}_{\vec{\tau}} = 0$$

$\vec{\mathcal{J}} = \vec{x} \times \vec{G} \rightarrow$ angular momentum density.

$\overleftrightarrow{\mathcal{K}} = -\overleftrightarrow{T} \times \vec{x} \rightarrow$ angular momentum flux tensor

29 Thus we have.

$$\frac{\partial}{\partial t}\vec{\mathcal{J}} + \vec{\nabla} \cdot \overleftrightarrow{\mathcal{K}} + \vec{\tau} = 0$$

IV Virial theorem

① Consider a particle in Newtonian mechanics in bounded motion

$$\vec{F} = m\frac{d\vec{v}}{dt}$$

② Averaging over time the quantity

$$\overline{(-\vec{F}\cdot\vec{r})} = -\int dt\ m\frac{d\vec{v}}{dt}\cdot\vec{r}$$

$$= -m\int dt\ \frac{d}{dt}(\vec{v}\cdot\vec{r}) + m\int dt\ \vec{v}\cdot\frac{d\vec{r}}{dt}$$

$$= -m\int dt\ \frac{d}{dt}(\vec{v}\cdot\vec{r}) + 2\int dt\left(\frac{1}{2}mv^2\right)$$

$$= -m\int dt\ \frac{d}{dt}(\vec{v}\cdot\vec{r}) + 2\overline{K}$$

③ Argue that boundary terms are irrelevant for a bounded motion, thus conclude.

$$-\overline{(\vec{F}\cdot\vec{r})} = 2\overline{K}$$

where bar means average over time. This is the general virial theorem.

(4) Consider the case when $\vec{F}$ is derived from a potential V

$$\vec{F} = -\vec{\nabla} V,$$

then (3) takes the form.

$$\vec{r} \cdot \vec{\nabla} \, \overline{V} = 2 \, \overline{K}.$$

(5) Consider the case when

$$V = k r^{\alpha}$$

then

$$\vec{r} \cdot \vec{\nabla} \, \overline{V} = r \frac{d}{dr} k r^{\alpha}$$

$$= \alpha \, \overline{V}$$

Thus, we have

$$\overline{K} = \frac{\alpha}{2} \overline{V}.$$

(6) For Coulomb potential, $V = -\frac{q_1 q_2}{4\pi\epsilon_0 r}$, $\alpha = -1$,

thus $\overline{K} = -\frac{1}{2} \overline{V}$.

7 Let us now evaluate the electromagnetic virial theorem. Using 27 (in page 9) we have. (taking dot product)

$$\frac{\partial}{\partial t}(\vec{r}\cdot\vec{G}) + \vec{\nabla}\cdot(\overleftrightarrow{T}\cdot\vec{r}) - \underbrace{Tr(\overleftrightarrow{T})}_{=U} + \vec{r}\cdot\vec{f} = 0$$

$$\frac{\partial}{\partial t}(\vec{r}\cdot\vec{G}) + \vec{\nabla}\cdot(\overleftrightarrow{T}\cdot\vec{r}) - U + \vec{r}\cdot\vec{f} = 0$$

This is the electromagnetic virial theorem.