SET-04

I Electrostatics and magnetostatics

(1) $\vec{\nabla}\cdot\vec{D} = \rho$ $\qquad$ $\vec{\nabla}\cdot\vec{B} = 0$ $\qquad$ $\vec{D} = \epsilon_0 \vec{E}$

$\vec{\nabla}\times E = -\frac{\partial}{\partial t}\vec{B}$ $\qquad$ $\vec{\nabla}\times\vec{H} = \frac{\partial}{\partial t}\vec{D} + \vec{J}$ $\qquad$ $\vec{H} = \frac{1}{\mu_0}\vec{B}$

and

$$\vec{F} = q\left[\vec{E} + \vec{v}\times\vec{B}\right]$$

(2) Statics is defined by

$$\frac{\partial\rho}{\partial t} = 0, \quad \frac{\partial\vec{J}}{\partial t} = 0, \quad \frac{\partial\vec{E}}{\partial t} = 0, \quad \frac{\partial\vec{B}}{\partial t} = 0.$$

For consistancy we require the statement of charge conservation to be

$$\cancel{\frac{\partial\rho}{\partial t}}^{=0} + \vec{\nabla}\cdot\vec{J} = 0$$

(3) Thus, we have.

Electrostatics

$$\vec{\nabla}\cdot\vec{D} = \rho$$
$$\vec{\nabla}\times\vec{E} = 0$$
$$\vec{D} = \epsilon_0\vec{E}$$

Magnetostatics

$$\vec{\nabla}\cdot\vec{B} = 0$$
$$\vec{\nabla}\times\vec{H} = \vec{J} \qquad (\Rightarrow \vec{\nabla}\cdot\vec{J} = 0)$$
$$\vec{H} = \frac{1}{\mu_0}\vec{B}$$

$\rightarrow$ $\vec{E}$ and $\vec{B}$ are decoupled.

II Uniqueness of solutions in electrostatics

① Maxwell's equations for electrostatics are.

$$\vec{\nabla} \cdot (\epsilon_0 \vec{E}) = \rho$$

$$\vec{\nabla} \times \vec{E} = 0 \quad \Rightarrow \quad \vec{E} = -\vec{\nabla} \phi$$

② Together, they read

$$\vec{\nabla} \cdot [\epsilon_0 \vec{\nabla} \phi] = \rho$$

③ What about boundary conditions? Let us show that $\vec{E} = 0$, at $r \to \infty$, is sufficient condition to require unique boundary conditions.

④ We shall prove this by contradiction. Let $\vec{E}_1$ and $\vec{E}_2$ be distinct (thus non-unique) solutions,

$$\vec{\nabla} \cdot \epsilon_0 \vec{E}_1 = \rho \qquad \vec{\nabla} \cdot \epsilon_0 \vec{E}_2 = \rho$$

$$\vec{\nabla} \times \vec{E}_1 = 0 \qquad \vec{\nabla} \times \vec{E}_2 = 0$$

⑤ Subtracting the equations gives.

$$\vec{\nabla} \cdot \epsilon_0 (\vec{E}_1 - \vec{E}_2) = 0$$

$$\vec{\nabla} \times (\vec{E}_1 - \vec{E}_2) = 0$$

(6) Let $\vec{\mathcal{E}} = \vec{E}_1 - \vec{E}_2$

$\mathcal{E}_0$ is position independent.

$$\vec{\nabla} \cdot \vec{\mathcal{E}} = 0$$

$$\vec{\nabla} \times \vec{\mathcal{E}} = 0$$

(7)
$$\underbrace{\vec{\nabla} \times (\vec{\nabla} \times \vec{\mathcal{E}})}_{=0} = \underbrace{\vec{\nabla}(\vec{\nabla} \cdot \vec{\mathcal{E}})}_{=0} - \nabla^2 \vec{\mathcal{E}}$$

$$\Rightarrow \quad \nabla^2 \vec{\mathcal{E}} = 0$$

(8) Let us consider one of the components

$$\nabla^2 \mathcal{E}_x = 0$$

$$\mathcal{E}_x \nabla^2 \mathcal{E}_x = 0$$

(since $\vec{E}_1$ & $\vec{E}_2$ are assumed to be distinct $\mathcal{E}_x$ is not zero.)

$$\vec{\nabla} \cdot (\mathcal{E}_x \vec{\nabla} \mathcal{E}_x) - (\vec{\nabla} \mathcal{E}_x)^2 = 0$$

$$\vec{\nabla} \cdot \vec{\nabla}\left(\frac{1}{2} \mathcal{E}_x^2\right) - (\vec{\nabla} \mathcal{E}_x)^2 = 0$$

$$\int_{V=\text{sphere}} d^3x \; \vec{\nabla} \cdot \vec{\nabla}\left(\frac{1}{2} \mathcal{E}_x^2\right) - \int d^3x \, (\vec{\nabla} \mathcal{E}_x)^2 = 0$$

$$4\pi R^2 \frac{d}{dR}\left(\frac{1}{2} \mathcal{E}_x^2\right) - \int d^3x \, (\vec{\nabla} \mathcal{E}_x)^2 = 0$$

$$\frac{d}{dR}\left(\frac{1}{2} \mathcal{E}_x\right)^2 - \frac{1}{4\pi R^2} \int d^3x \, (\vec{\nabla} \mathcal{E}_x)^2 = 0$$

$$\int_0^\infty dR \frac{d}{dR} \frac{1}{2} \mathcal{E}_x^2 - \frac{1}{4\pi} \int_0^\infty dR \frac{1}{R^2} \int d^3x \, (\vec{\nabla} \mathcal{E}_x)^2 = 0$$

$$\frac{1}{2} \mathcal{E}_x^2 \Big|_{\text{at } R=\infty} - \underbrace{\frac{1}{2} \mathcal{E}_x^2 \Big|_{\text{at } R=0}}_{+} - \frac{1}{4\pi} \underbrace{\int_0^\infty dR \frac{1}{R^2} \int d^3x \, (\vec{\nabla} \mathcal{E}_x)^2}_{+} = 0$$

(9) If $\mathcal{E}_x$ at $R \to \infty$ is 0, then two positive quantities can be zero only if each is zero.

Thus,

(i) $\mathcal{E}_x(0) = 0$

(ii) $\vec{\nabla} \mathcal{E}_x = 0$, everywhere.

(10) Since, our choice of origin of sphere was arbitrary, we also have.

$\mathcal{E}_x(0)$, everywhere.

(11) Thus, we have contradicted the first assumption

$$\vec{\mathcal{E}} = \vec{E}_1 - \vec{E}_2 = 0,$$

which implies that solution to electrostatics is unique.

$\longrightarrow$ Also refer to problems in Schwinger et al., chapter 1.

III Earnshaw's theorem

① $$\vec{\nabla}\cdot \epsilon_0 \vec{E} = \rho$$
$$\vec{\nabla}\times \vec{E} = 0$$

② Let us investigate if the electric field created by ρ at another point ($\rho = 0$) can stabilize a charge.

③ In the region where $\rho = 0$ we have.

$$\vec{\nabla}\cdot \vec{E} = 0$$
$$\vec{\nabla}\times \vec{E} = 0$$

④ $$\underbrace{\vec{\nabla}\times(\vec{\nabla}\times\vec{E})}_{=0} = \vec{\nabla}(\underbrace{\vec{\nabla}\cdot\vec{E}}_{=0}) - \nabla^2\vec{E}$$

$$\Rightarrow \quad \nabla^2 \vec{E} = 0$$

⑤ $$f(x), \quad \frac{d}{dx} f = 0, \quad \frac{d^2 f}{dx^2} \begin{cases} > 0 & \text{minima} \\ < 0 & \text{maxima} \\ = 0 & \text{saddle pt.} \end{cases}$$

⑥ Since each component of $\vec{E}$ satisfies $\vec{\nabla} E_x = 0$, we can not have stability.

⑦ Seemingly contradictory examples:

- Levitron → spinning top (not static)
- Levitating frog (diamagnetism)
- Levitating magnets near surfaces (presence of boundaries).
- Can Stark effect stabilize ions (see work in Univ of Oklahoma Physics department.)

IV Potential for a point charge.

1. Charge density for a point charge is

$$\rho(\vec{r}) = q\,\delta^{(3)}(\vec{r} - \vec{r}_a)$$

$\vec{r}$ – space variable
$\vec{r}_a$ – position of charge.

2. From electrostatic Maxwell's equations we have

$$-\nabla^2 \phi(\vec{r}) = \frac{q}{\epsilon_0}\,\delta^{(3)}(\vec{r} - \vec{r}_a)$$

3. Fourier transformation in 1-D.

$$f(x) = \int_{-\infty}^{+\infty} \frac{dk}{2\pi}\, e^{ikx}\, \tilde{f}(k)$$

$$\tilde{f}(k) = \int_{-\infty}^{+\infty} dx\, e^{-ikx}\, f(x)$$

$$\delta(x) = \int_{-\infty}^{+\infty} \frac{dk}{2\pi}\, e^{ikx} \cdot 1.$$

$$1 = \int_{-\infty}^{+\infty} dx\, e^{-ikx}\, \delta(x).$$

4. Let (generalizing it to 3-D)

$$\phi(\vec{r}) = \int \frac{d^3k}{(2\pi)^3}\, e^{i\vec{k}\cdot(\vec{r} - \vec{r}_a)}\, \tilde{\phi}(\vec{k})$$

5. Using 4 in 2

$$k^2\, \tilde{\phi}(\vec{k}) = \frac{q}{\epsilon_0}$$

$$\tilde{\phi}(\vec{k}) = \frac{q}{\epsilon_0}\,\frac{1}{k^2}.$$

→ $\vec{\nabla} e^{i\vec{k}\cdot\vec{R}} = i\vec{k}\, e^{i\vec{k}\cdot\vec{R}}$

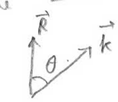
→ $e^{i\vec{k}\cdot\vec{R}}$ forms a complete set.

6

$$\phi(\vec r) = \int \frac{d^3k}{(2\pi)^3}\, e^{i\vec k\cdot(\vec r-\vec r_a)}\, \tilde\phi(\vec k)$$

$\vec r - \vec r_a = \vec R$

choose $\vec R$ along the z-direction.



$$= \int \frac{d^3k}{(2\pi)^3}\, e^{i\vec k\cdot(\vec r-\vec r_a)}\, \frac{q}{\epsilon_0}\,\frac{1}{k^2}$$

$$= \frac{1}{(2\pi)^3}\int_0^\infty k^2\,dk \int_0^\pi \sin\theta\, d\theta \cdot \int_0^{2\pi} d\phi\; e^{ikR\cos\theta}\,\frac{q}{\epsilon_0}\,\frac{1}{k^2}$$

$$= \frac{1}{(2\pi)^3}\int_0^\infty dk \int_0^\pi \sin\theta\, d\theta\; e^{ikR\cos\theta}\; 2\pi\,\frac{q}{\epsilon_0}$$

$t = \cos\theta$

$dt = -\sin\theta\, d\theta.$

$$= \frac{q}{4\pi\epsilon_0}\,\frac{1}{\pi}\int_0^\infty dk \int_{-1}^{+1} dt\; e^{ikRt}$$

$$= \frac{q}{4\pi\epsilon_0}\,\frac{1}{\pi}\int_0^\infty dk\,\frac{1}{ikR}\left(e^{ikR} - e^{-ikR}\right)$$

$$= \frac{q}{4\pi\epsilon_0}\,\frac{2}{\pi}\int_0^\infty \frac{dk}{kR}\,\sin kR$$

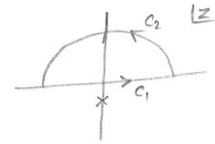
$$= \frac{q}{4\pi\epsilon_0}\,\frac{1}{R}\,\frac{2}{\pi}\int_0^\infty \frac{dx}{x}\,\sin x.$$

$$\int_0^\infty \frac{dx}{x}\,\sin x = \frac{\pi}{2}$$

$$= \frac{q}{4\pi\epsilon_0}\,\frac{1}{R}$$

$$= \frac{q}{4\pi\epsilon_0}\,\frac{1}{|\vec r - \vec r_a|}$$

7 $\quad I = \int_{-\infty}^{+\infty} \frac{dx}{x+i\epsilon}\, e^{ix}.$



Consider the complex integral $\int_C \frac{dz}{z+i\epsilon}\, e^{iz}.$

$\underline{\epsilon > 0}$:

$$\int_{C_1} \frac{dx}{x+i\epsilon}\, e^{ix} + \int_{C_2} \frac{dz}{z+i\epsilon}\, e^{iz} = 0 \qquad z = Re^{i\theta}$$

$$I + \int_0^{\pi} \frac{iRe^{i\theta}\, d\theta}{(Re^{i\theta} + i\epsilon)}\, e^{iR\cos\theta - R\sin\theta} = 0$$

for large R we have.

$$I + \int_0^{\pi} \frac{i\,d\theta}{1}\, \underbrace{e^{-R\sin\theta}}_{\to 0} = 0$$

$$I = 0.$$

$\underline{\epsilon < 0}$: Pole contributes and we will get

$$I = 2\pi i\, e^{\epsilon}$$

Thus we have.

$$I = \int_{-\infty}^{+\infty} \frac{dx}{x+i\epsilon}\, e^{ix} = \begin{cases} 0 & \text{if } \epsilon > 0 \\ 2\pi i\, e^{\epsilon} & \text{if } \epsilon < 0 \end{cases}$$

8 $\quad$ Taking the complex conjugate we have.

$$I^* = \int_{-\infty}^{+\infty} \frac{dx}{x-i\epsilon}\, e^{-ix} = \begin{cases} 0 & \text{if } \epsilon > 0 \\ -2\pi i\, e^{\epsilon} & \text{if } \epsilon < 0. \end{cases}$$

(9) Using (8) we can conclude, by switching the sign of ϵ,

$$\int_{-\infty}^{+\infty} \frac{dx}{x+i\epsilon}\, e^{-ix} = \begin{cases} -2\pi i\, e^{-\epsilon} & \text{if } \epsilon > 0 \\ 0 & \text{if } \epsilon < 0 \end{cases}$$

(10) Together we have.

$$\int_{-\infty}^{+\infty} \frac{dx}{x+i\epsilon}\, e^{ix} = \begin{cases} 0 & \text{if } \epsilon > 0 \\ 2\pi i\, e^{i\epsilon} & \text{if } \epsilon < 0 \end{cases}$$

$$\int_{-\infty}^{+\infty} \frac{dx}{x+i\epsilon}\, e^{-ix} = \begin{cases} -2\pi i\, e^{-\epsilon} & \text{if } \epsilon > 0 \\ 0 & \text{if } \epsilon < 0 \end{cases}$$

(11) Subtracting the two expressions we have.

$$\int_{-\infty}^{+\infty} \frac{dx}{x+i\epsilon}\, (e^{ix} - e^{-ix}) = \begin{cases} 2\pi i\, e^{-\epsilon} & \text{if } \epsilon > 0 \\ 2\pi i\, e^{\epsilon} & \text{if } \epsilon < 0 \end{cases}$$

We can get rid of the ϵ dependence now by taking the limit $\epsilon \to 0$.

$$\int_{-\infty}^{+\infty} \frac{dx}{x}\, (e^{ix} - e^{-ix}) = 2\pi i, \quad \text{everywhere.}$$

(12) Thus,

$$\int_{-\infty}^{+\infty} \frac{dx}{x} \sin x = \pi$$