I Maxwell's equations in various units

1 SI units

$$\vec{\nabla}\cdot\vec{D} = \rho$$

$$\vec{\nabla}\cdot\vec{B} = 0$$

$$-\vec{\nabla}\times\vec{E} = \frac{\partial}{\partial t}\vec{B}\,.$$

$$\vec{\nabla}\times\vec{H} = \frac{\partial}{\partial t}\vec{D} + \vec{J}$$

$$\vec{D} = \epsilon_0\vec{E} + \vec{P}$$

$$\vec{H} = \frac{1}{\mu_0}\vec{B} - \vec{M}$$

$$\mu_0 = 4\pi\times10^{-7}\ \frac{\text{Tesla}\cdot\text{meter}}{\text{Ampere}}\,.$$

$$c = 299\,729\,458\ \text{m/s (exact)}$$

$$\epsilon_0\mu_0 = \frac{1}{c^2}\,.$$

Lorentz force law

$$\vec{F} = q\left[\vec{E} + \vec{v}\times\vec{B}\right]$$

2 Gaussian units

$$\vec{\nabla}\cdot\vec{D} = 4\pi\rho$$

$$\vec{\nabla}\cdot\vec{B} = 0$$

$$-\vec{\nabla}\times\vec{E} = \frac{1}{c}\frac{\partial}{\partial t}\vec{B}\,.$$

$$\vec{\nabla}\times\vec{H} = \frac{1}{c}\frac{\partial}{\partial t}\vec{D} + \frac{4\pi}{c}\vec{J}$$

$$\vec{D} = \vec{E} + 4\pi\vec{P}$$

$$\vec{H} = \vec{B} - 4\pi\vec{M}$$

Lorentz force law

$$\vec{F} = q\left[\vec{E} + \frac{\vec{v}}{c}\times\vec{B}\right]$$

(3) Lorentz-Heaviside units

$$\vec{\nabla}\cdot\vec{D} = \rho$$

$$\vec{\nabla}\cdot\vec{B} = 0$$

$$-\vec{\nabla}\times\vec{E} = \frac{\partial}{\partial t}\vec{B}$$

$$\vec{\nabla}\times\vec{H} = \frac{\partial}{\partial t}\vec{D} + \vec{J}$$

$$\vec{D} = \vec{E} + \vec{P}$$

$$\vec{H} = \vec{B} - \vec{M}$$

Lorentz force law

$$\vec{F} = q\left[\vec{E} + \frac{\vec{v}}{c}\times\vec{B}\right]$$

(4) Conversions

$$\vec{D}_G = \sqrt{\frac{4\pi}{\epsilon_0}}\,\vec{D}_{SI} = \sqrt{4\pi}\,\vec{D}_{LH}$$

$$\vec{E}_G = \sqrt{4\pi\epsilon_0}\,\vec{E}_{SI} = \sqrt{4\pi}\,\vec{E}_{LH}$$

$$\vec{H}_G = \sqrt{4\pi\mu_0}\,\vec{H}_{SI} = \sqrt{4\pi}\,\vec{H}_{LH}$$

$$\vec{B}_G = \sqrt{\frac{4\pi}{\mu_0}}\,\vec{B}_{SI} = \sqrt{4\pi}\,\vec{B}_{LH}$$

$$\rho_G = \frac{1}{\sqrt{4\pi\epsilon_0}}\,\rho_{SI} = \frac{1}{\sqrt{4\pi}}\,\rho_{LH}$$

$$\vec{P}_G = \frac{1}{\sqrt{4\pi\epsilon_0}}\,\vec{P}_{SI} = \frac{1}{\sqrt{4\pi}}\,\vec{P}_{LH}$$

$$\vec{J}_G = \frac{1}{\sqrt{4\pi\epsilon_0}}\,\vec{J}_{SI} = \frac{1}{\sqrt{4\pi}}\,\vec{J}_{LH}$$

$$\vec{M}_G = \sqrt{\frac{\mu_0}{4\pi}}\,\vec{M}_{SI} = \frac{1}{\sqrt{4\pi}}\,\vec{M}_{LH}$$

II Statement of conservation of charge

1. In SI unit we have.

(i) $\vec{\nabla} \cdot \vec{D} = \rho$

(ii) $\vec{\nabla} \cdot \vec{B} = 0$

(iii) $-\vec{\nabla} \times \vec{E} = \frac{\partial}{\partial t} \vec{B}$

(iv) $\vec{\nabla} \times \vec{H} = \frac{\partial}{\partial t} \vec{D} + \vec{J}$

$\vec{D} = \epsilon_0 \vec{E}$

$\vec{H} = \frac{1}{\mu_0} \vec{B}$

$\vec{F} = q\,[\vec{E} + \vec{v} \times \vec{B}]$.

2. Using 1-(i)

$$\frac{\partial}{\partial t} \vec{\nabla} \cdot \vec{D} = \frac{\partial \rho}{\partial t}$$

3. Using 1-(iv)

$$\vec{\nabla} \cdot \vec{\nabla} \times \vec{H} = \vec{\nabla} \cdot \frac{\partial}{\partial t} \vec{D} + \vec{\nabla} \cdot \vec{J}$$

$$0 = \vec{\nabla} \cdot \frac{\partial}{\partial t} \vec{D} + \vec{\nabla} \cdot \vec{J}$$

← Prove explicitly $\vec{\nabla} \cdot \vec{\nabla} \times \vec{H} = 0$.

$$\epsilon_{ijk} \nabla_j \nabla_k = \epsilon_{ijk} [\nabla_j \nabla_k + \nabla_k \nabla_j] \tfrac{1}{2}$$

$$= (\epsilon_{ijk} - \epsilon_{ikj}) \tfrac{1}{2} \nabla_j \nabla_k$$

$$= 0$$

4. Using 2 & 3

$$\frac{\partial \rho}{\partial t} + \vec{\nabla} \cdot \vec{J} = 0$$

5. Integrating 4 over a volume bounded by surface S.

$$\frac{\partial}{\partial t} \underbrace{\int_V d^3 r \, \rho(\vec{r}, t)}_{Q(t)} + \int_V d^3 r \, \vec{\nabla} \cdot \vec{J} = 0$$

$$\frac{\partial}{\partial t} Q(t) + \oint d\vec{a} \cdot \vec{J} = 0$$

III Electric and magnetic potential

(1) $\vec{\nabla} \cdot \vec{B} = 0$

A sufficient condition for the solution is

$\vec{B} = \vec{\nabla} \times \vec{A}$ $\quad (\because \vec{\nabla} \cdot \vec{\nabla} \times \vec{A} = 0)$

(2) Is $\vec{A}(\vec{x}, t)$ unique? No.

(3) Using $\vec{B} = \vec{\nabla} \times \vec{A}$ in the other source-less (homogeneous) equation we get

$$\vec{\nabla} \times \vec{E} + \frac{\partial}{\partial t}(\vec{\nabla} \times \vec{A}) = 0$$

$$\vec{\nabla} \times \Big[\underbrace{\vec{E} + \frac{\partial \vec{A}}{\partial t}}_{-\vec{\nabla}\phi} \Big] = 0$$

(4) Again, a sufficient requirement is:

$\vec{E} + \frac{\partial \vec{A}}{\partial t} = -\vec{\nabla}\phi$ $\quad (\because \vec{\nabla} \times \vec{\nabla}\phi = 0)$

(5) → the negative sign in (4) is a conventional choice.

→ Is $\phi(\vec{x}, t)$ unique? No.

(6) Thus, we have

$\vec{B} = \vec{\nabla} \times \vec{A}$

$\vec{E} = -\vec{\nabla}\phi - \frac{\partial}{\partial t}\vec{A}$

$\vec{A}(\vec{x}, t) \rightarrow$ magnetic vector potential

$\phi(\vec{x}, t) \rightarrow$ electric scalar potential

IV Gauge transformations (non-uniqueness of ϕ and $\vec{A}$)

1. — $\vec{E}$ and $\vec{B}$ are physical quantities
   — ϕ and $\vec{A}$ are mathematical constructs.
   — Quantum mechanics is based on the Hamiltonian, which is written in terms of ϕ and $\vec{A}$.
   — Ahranov - Bohm effect (1959) suggests that ϕ and $\vec{A}$ might have experimental signatures.

2. Let
$$\left.\begin{aligned} \vec{A}' &= \vec{A} + \vec{\nabla}\lambda(\vec{x},t) \\ \phi' &= \phi - \frac{\partial}{\partial t}\lambda(\vec{x},t) \end{aligned}\right\} \text{— Gauge transformation.}$$

3.
$$\begin{aligned} \vec{B}' = \vec{\nabla}\times\vec{A}' &= \vec{\nabla}\times\left[\vec{A} + \vec{\nabla}\lambda\right] \\ &= \vec{\nabla}\times\vec{A} + \vec{\nabla}\times\vec{\nabla}\lambda \\ &= \vec{B} \end{aligned}$$
$$\vec{\nabla}\times\vec{\nabla} = 0$$

4.
$$\begin{aligned} \vec{E}' &= -\vec{\nabla}\phi' - \frac{\partial}{\partial t}\vec{A}' \\ &= -\vec{\nabla}\left[\phi - \frac{\partial}{\partial t}\lambda\right] - \frac{\partial}{\partial t}\left[\vec{A} + \vec{\nabla}\lambda\right] \\ &= -\vec{\nabla}\phi - \frac{\partial}{\partial t}\vec{A} = \vec{E} \end{aligned}$$

5. Thus, physical quantities ($\vec{E}$ and $\vec{B}$) do not change under gauge transformation of ②. Gauge dependence is an artifact of the mathematical construct involved in introducing ϕ and $\vec{A}$.

6. We will later show how gauge invariance is connected to conservation of charge.

VI Choice of gauge

(1)
$$-\nabla^2\phi - \frac{\partial}{\partial t}(\vec{\nabla}\cdot\vec{A}) = \frac{\rho}{\epsilon_0}$$

$$-\nabla^2\vec{A} + \frac{1}{c^2}\frac{\partial^2}{\partial t^2}\vec{A} + \vec{\nabla}\left[\vec{\nabla}\cdot\vec{A} + \frac{1}{c^2}\frac{\partial}{\partial t}\phi\right] = \mu_0\vec{J}$$

(2) Lorentz gauge is the choice.

$$\vec{\nabla}\cdot\vec{A} + \frac{1}{c^2}\frac{\partial}{\partial t}\phi = 0,$$

which used in (1) gives.

$$-\nabla^2\phi + \frac{1}{c^2}\frac{\partial^2}{\partial t^2}\phi = \frac{\rho}{\epsilon_0}$$

$$-\nabla^2\vec{A} + \frac{1}{c^2}\frac{\partial^2}{\partial t^2}\vec{A} = \mu_0\vec{J}$$

and are inhomogenous (with source) wave equation.

(3) Coulomb gauge is the choice.

$$\vec{\nabla}\cdot\vec{A} = 0,$$

which used in (1) gives.

$$-\nabla^2\phi = \frac{\rho}{\epsilon_0}$$

$$-\nabla^2\vec{A} + \frac{1}{c^2}\frac{\partial^2}{\partial t^2}\vec{A} = \mu_0\vec{J} - \frac{1}{c^2}\frac{\partial}{\partial t}\vec{\nabla}\phi.$$

This has the advantage that determination of ϕ does not contain time dependence.