SET - 02

1 Summation convention

Consider a vector transformation

eg: rotation

$$\begin{pmatrix} A_1' \\ A_2' \\ A_3' \end{pmatrix} = \begin{pmatrix} R_{11} & R_{12} & R_{13} \\ R_{21} & R_{22} & R_{23} \\ R_{31} & R_{32} & R_{33} \end{pmatrix} \begin{pmatrix} A_1 \\ A_2 \\ A_3 \end{pmatrix}$$

$$A_1' = R_{11} A_1 + R_{12} A_2 + R_{13} A_3$$

$$= \sum_{i=1}^{3} R_{1j} A_j$$

(summation convention)

$$= R_{1j} A_j$$

$$A_i = R_{ij} A_j$$

2 For matrix multiplication

$$\overleftrightarrow{A} \cdot \overleftrightarrow{B} = \overleftrightarrow{C}$$

we write

$$A_{ij} B_{jk} = C_{ik}$$

3 $$\overleftrightarrow{1} \longrightarrow \delta_{ij} = \begin{cases} 1 & \text{if } i = j \\ 0 & \text{if } i \neq j \end{cases}$$

(2) Conserved quantities under rotation

- length of a vector (any dot product)
- angle between two vectors.

(3) Length of a vector

{dummy index versus free index}.

$$\vec{x}\cdot\vec{x} = x_i x_i = x_i \delta_{ij} x_j = x_m \delta_{mn} x_n$$

$$\vec{x}'\cdot\vec{x}' = x'_i x'_i = R_{im} x_m R_{in} x_n.$$

$$\Rightarrow \quad R_{im} R_{in} = \delta_{mn}.$$

$$(R^T)_{mi} R_{in} = \delta_{mn}$$

$$\overleftrightarrow{R}^T \cdot \overleftrightarrow{R} = \overleftrightarrow{1}$$

$\Rightarrow$ the transformation that leaves the length of a vector invariant is an orthogonal transformation.

(4) Quantities that transform like a vector is called a vector

$$x'_i = R_{ij} x_j$$

Scalar $\longrightarrow$ $f(x'_i) = f(x_i)$

Vector $\longrightarrow$ $A'_i = R_{ij} A_j$

2nd rank tensor $\rightarrow$ $A'_{ij} = R_{im} R_{jn} A_{mn}$.

nth rank tensor $\rightarrow$ $A'_{i_1 \ldots i_n} = R_{i_1 j_1} R_{i_2 j_2} \cdots R_{i_n j_n} A_{j_1 \ldots j_n}$

5. Kronecker delta

$$\delta_{ij} = \begin{cases} 1 & \text{if } i=j \\ 0 & \text{if } i\neq j \end{cases}$$

6. Levi-Civita symbol

$$\epsilon_{ijk} = \begin{cases} +1 & \text{if } i=1,\ j=2,\ k=3 \text{ and even permutations} \\ -1 & \text{if } i=1,\ j=3,\ k=2 \text{ and even permutation} \\ 0 & \text{otherwise} \end{cases}$$

7.
$$\epsilon_{ijk}\,\epsilon_{pqr} = \begin{vmatrix} \delta_{ip} & \delta_{iq} & \delta_{ir} \\ \delta_{jp} & \delta_{jq} & \delta_{jr} \\ \delta_{kp} & \delta_{kq} & \delta_{kr} \end{vmatrix}$$

8. Also note $\delta_{ii} = 3$

9. We can show

$$\epsilon_{ijk}\,\epsilon_{iqr} = \delta_{jq}\,\delta_{kr} - \delta_{jr}\,\delta_{kq}.$$

$$\epsilon_{ijk}\,\epsilon_{ijr} = 2\,\delta_{kr}$$

$$\epsilon_{ijk}\,\epsilon_{ijk} = 6$$

(10) <u>Verify</u>:

(i) $\vec{A}\cdot\vec{B} = A_i B_i$

(ii) $(\vec{A}\times\vec{B})_i = \epsilon_{ijk} A_j B_k$

(iii) $\mathrm{Tr}(\overleftrightarrow{M}) = M_{ii}$

(iv) $\det(\overleftrightarrow{M}) = \epsilon_{ijk} M_{i1} M_{j2} M_{k3}$

$$= \frac{1}{3!}\epsilon_{ijk}\epsilon_{i'j'k'} M_{ii'} M_{jj'} M_{kk'}$$

(11) <u>Examples</u>:

$$\vec{A}\times(\vec{B}\times\vec{C}) = \vec{B}(\vec{A}\cdot\vec{C}) - \vec{C}(\vec{A}\cdot\vec{B})$$

$$(\vec{A}\times\vec{B})\cdot(\vec{C}\times\vec{D}) = (\vec{A}\cdot\vec{C})(\vec{B}\cdot\vec{D}) - (\vec{A}\cdot\vec{D})(\vec{B}\cdot\vec{C})$$

(12) Vector calculus

$$f = f(x) \qquad df = dx \frac{\partial f}{\partial x}.$$

$$f = f(x,y,z) \qquad df = dx \frac{\partial f}{\partial x} + dy \frac{\partial f}{\partial y} + dz \frac{\partial f}{\partial z}.$$

(13)

$$d\vec{l} = dx\, \hat{i} + dy\, \hat{j} + dz\, \hat{k}.$$

$$\vec{\nabla} = \hat{i} \frac{\partial}{\partial x} + \hat{j} \frac{\partial}{\partial y} + \hat{k} \frac{\partial}{\partial x}.$$

(14)

$$d\vec{l} \rightarrow dl_i$$

$$\vec{\nabla} \rightarrow \nabla_i$$

(15)

$$\begin{aligned}
\vec{\nabla} \times (\vec{A} \times \vec{B}) &= \epsilon^{ijk} \nabla^j \epsilon^{kmn} A^m B^n. \\
&= (\delta^{im}\delta^{jn} - \delta^{in}\delta^{jm}) \nabla^j (A^m B^n) \\
&= (\delta^{im}\delta^{jn} - \delta^{in}\delta^{jm}) \left[(\nabla^j A^m) B^n + A^m \nabla^j B^n \right] \\
&= (\nabla^j A^i) B^j + A^i \nabla^j B^j - (\nabla^j A^j) B^i - A^j \nabla^j B^i \\
&= (\vec{B} \cdot \vec{\nabla}) \vec{A} + \vec{A} (\vec{\nabla} \cdot \vec{B}) - (\vec{\nabla} \cdot \vec{A}) \vec{B} - (\vec{A} \cdot \vec{\nabla}) \vec{B}.
\end{aligned}$$

(16) $$\vec{\nabla}\, r = \vec{\nabla}\sqrt{x^2+y^2+z^2} = \hat{r} = \frac{\vec{r}}{r}$$

$$\vec{\nabla}\, \vec{r} = \overleftrightarrow{1}$$

(18) $$\vec{\nabla}\cdot\frac{\vec{r}}{r^3} \overset{?}{=} 0$$

$$\int d^3r\ \vec{\nabla}\cdot\frac{\vec{r}}{r^3} = \oint_R d\vec{a}\cdot\frac{\vec{r}}{r^3}$$

$$= 4\pi R^2\ \frac{\vec{R}}{R}\cdot\frac{\vec{R}}{R^3}$$

$$= 4\pi$$

surface of integration

Argue that

$$\vec{\nabla}\cdot\frac{\vec{r}}{r^3} = 4\pi\ \delta^{(3)}(\vec{r})\,.$$

19 Delta function

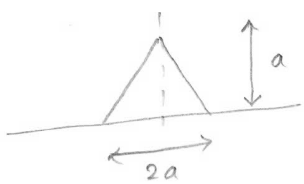
$$\delta(x) = \begin{cases} 0 & x \neq 0 \\ \infty & x = 0 \end{cases}$$

and satisfies

$$\int_{-\infty}^{+\infty} dx\, f(x)\, \delta(x) = f(0)$$

for a "well defined" function $f(x)$.

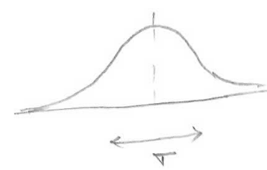
20 Visual example :



$$\text{Area} = \frac{1}{2}\, 2a\, \frac{1}{a} = 1$$

$$\underset{a \to 0}{Lt}\, (\text{Area}) = 1$$

21 $$\delta(x) = \underset{\sigma \to 0}{Lt}\, \frac{1}{\sqrt{2\pi\sigma}}\, e^{-\frac{x^2}{2\sigma}}$$



$$\int_{-\infty}^{+\infty} dx\, \delta(x) = \underset{\sigma \to 0}{Lt}\, \frac{1}{\sqrt{2\pi\sigma}} \int_{-\infty}^{+\infty} dx\, e^{-\frac{x^2}{2\sigma}}$$

$$= \underset{\sigma \to 0}{Lt}\, \frac{1}{\sqrt{2\pi\sigma}} \sqrt{\pi\, 2\sigma}$$

$$= 1 .$$

22 $\delta(ax) = \frac{1}{a}\delta(x)$ $\qquad y = ax.$

$$\int_{-\infty}^{+\infty} dx\, f(x)\, \delta(ax) = \int_{-\infty}^{+\infty} \frac{dy}{a}\, f\left(\frac{y}{a}\right) \delta(y)$$

$$= \frac{1}{a} f(0)$$

$$\int_{-\infty}^{+\infty} dx\, f(x)\, \delta(x) = f(0)$$

$$\Rightarrow \quad \delta(ax) = \frac{1}{a}\delta(x)$$

23 $$\delta(g(x)) = \sum_i \frac{\delta(x-a_i)}{\left|\frac{\partial g}{\partial x}\right|_{x=a_i}}$$

where it is understood that we can write

$$g(x) = g_0 (x-a_1)(x-a_2)\cdots(x-a_n)$$

Show this for $g(x) = (x-a)(x-b)$