(Optional) Homework No. 06 (Fall 2013)

PHYS 520A: Electromagnetic Theory I

Due date: No submission required.

A forced harmonic oscillator is described by the differential equation

$$-\left(\frac{d^2}{dt^2}+\omega^2\right)x(t)=F(t), \tag{1}$$

with appropriate initial conditions, say,

$$x(0)=-A, \qquad \text{and} \qquad \dot{x}(0)=\left.\frac{dx(t)}{dt}\right|_{t=0}=0. \tag{2}$$

Here ω is the angular frequency of the oscillator and $F(t)$ is a priori given forcing function (or the source). The corresponding Green's function satisfies

$$-\left(\frac{d^2}{dt^2}+\omega^2\right)G(t,t')=\delta(t-t'). \tag{3}$$

1. Show that the solution, $x(t)$, to the differential equation in Eq. (1), is given in terms of the Greens function by

$$\begin{aligned} x(t) &= \int_{-\infty}^{+\infty} dt' G(t,t')F(t') + \int_{-\infty}^{+\infty} dt' \frac{d}{dt'}\left[\left(x(t') - x(t')\frac{d}{dt'}\right)G(t,t')\right] && (4)\\ &= \int_{-\infty}^{+\infty} dt' G(t,t')F(t') \\ &\quad + \lim_{\tau_2\to+\infty}\left[x(\tau_2)-x(\tau_2)\frac{d}{d\tau_2}\right]G(t,\tau_2) - \lim_{\tau_1\to-\infty}\left[x(\tau_1)-x(\tau_1)\frac{d}{d\tau_1}\right]G(t,\tau_1), && (5)\end{aligned}$$

   where the limiting variables in the second equality are constructed such that $\tau_1 < \{t,t'\} < \tau_2$.

2. The corresponding homogeneous differential equation is

$$-\left(\frac{d^2}{dt^2}+\omega^2\right)x_0(t)=0. \qquad \text{and} \qquad -\left(\frac{d^2}{dt^2}+\omega^2\right)G_0(t,t')=0. \tag{6}$$

   (a) Show that for a Greens function, $G(t,t')$, that solves Eq. (3),

$$G(t,t')+G_0(t,t') \tag{7}$$

   is also a solution to Eq. (3).

(b) Show that the homogeneous solution of the Greens function does not contribute to $x(t)$ by showing that

$$\int_{-\infty}^{+\infty} dt' G_0(t,t') F(t') + \int_{-\infty}^{+\infty} dt' \frac{d}{dt'} \left[\left(x(t') - x(t') \frac{d}{dt'} \right) G_0(t,t') \right] = 0. \tag{8}$$

(c) Argue that the surface terms in Eq. (3) satisfy the homogeneous differential equation in Eq. (6)

$$-\left(\frac{d^2}{dt^2} + \omega^2 \right) \left[\left(\dot{x}(\tau) - x(\tau) \frac{d}{d\tau} \right) \bar{G}(t,\tau) \right] = 0, \tag{9}$$

because the surface points, denoted by τ above, never equals the variable t, i.e. $\tau \neq t$.

3. Beginning with Eq. (3) derive the continuity conditions satisfied by the Greens function at $t = t'$ to be

$$\frac{d}{dt} G(t,t') \Big|_{t=t'-\delta}^{t=t'+\delta} = -1 \tag{10}$$

and

$$G(t,t') \Big|_{t=t'-\delta}^{t=t'+\delta} = 0. \tag{11}$$

4. For all points, except $t = t'$, the differential Eq. (3) has no source term and thus reads like the equation for $G_0(t,t')$ in Eq.(6). This equation has oscillatory solutions, which could have different behavior at $t < t'$ and $t > t'$, except for the constraint imposed by the continuity conditions in Eqs. (10) and (11). In terms of four arbitrary functions of t', A, B, C, and D, we can write

$$G(t,t') = \begin{cases} A(t')\, e^{i\omega t} + B(t')\, e^{-i\omega t}, & \text{if } t < t', \\ C(t')\, e^{i\omega t} + D(t')\, e^{-i\omega t}, & \text{if } t > t'. \end{cases} \tag{12}$$

Imposing the continuity conditions in Eqs. (10) and (11) derive the following equations constraining $A(t')$, $B(t')$, $C(t')$, and $D(t')$:

$$[C(t') - A(t')]\, e^{i\omega t'} + [D(t') - B(t')]\, e^{-i\omega t'} = 0, \tag{13}$$

$$[C(t') - A(t')]\, e^{i\omega t'} - [D(t') - B(t')]\, e^{-i\omega t'} = \frac{i}{\omega}. \tag{14}$$

5. Using the continuity conditions and without imposing any boundary conditions solve for $G(t,t')$ in the following four forms:

$$\begin{aligned} G(t,t') &= A(t')\, e^{i\omega t} + B(t')\, e^{-i\omega t} + G_R(t-t') && \text{(15a)} \\ &= C(t')\, e^{i\omega t} + D(t')\, e^{-i\omega t} + G_A(t-t') && \text{(15b)} \\ &= A(t')\, e^{i\omega t} + D(t')\, e^{-i\omega t} + G_F(t-t') && \text{(15c)} \\ &= C(t')\, e^{i\omega t} + B(t')\, e^{-i\omega t} + G_W(t-t') && \text{(15d)} \end{aligned}$$

where

$$G_R(t-t') = -\frac{1}{\omega}\,\theta(t-t')\,\sin\omega(t-t'), \tag{16a}$$

$$G_A(t-t') = +\frac{1}{\omega}\,\theta(t'-t)\,\sin\omega(t-t'), \tag{16b}$$

$$G_F(t-t') = -\frac{1}{\omega}\frac{1}{2i}\left[+\theta(t-t')\,e^{i\omega(t-t')} + \theta(t'-t)\,e^{-i\omega(t-t')}\right], \tag{16c}$$

$$G_W(t-t') = -\frac{1}{\omega}\frac{1}{2i}\left[-\theta(t'-t)\,e^{i\omega(t-t')} - \theta(t-t')\,e^{-i\omega(t-t')}\right], \tag{16d}$$

and the subscripts stand for retarded, advanced, Feynman, and Wheeler, respectively. Recognize that the above four forms are special cases of the following general expression

$$-\frac{1}{\omega}\frac{1}{2i}\left[a\,\theta(t-t') - b\,\theta(t'-t)\right] e^{i\omega(t-t')} + \frac{1}{\omega}\frac{1}{2i}\left[c\,\theta(t-t') - d\,\theta(t'-t)\right] e^{-i\omega(t-t')}, \tag{17}$$

where the numerical constants a, b, c, and d, are arbitrary to the extent that they obey the constraints $a+b=1$, and $c+d=1$. The special cases, $a=1$, $c=1$, corresponds to G_R; $a=0$, $c=0$, corresponds to G_A; $a=1$, $c=0$, corresponds to G_F; and $a=0$, $c=1$, corresponds to G_W, respectively.

6. Show that we can write

$$x(t) = \int_{-\infty}^{+\infty} dt' G(t-t')F(t') + \alpha_0\,e^{i\omega t} + \beta_0\,e^{-i\omega t}, \tag{18}$$

where α_0 and β_0 are the arbitrary numerical constants. Use the initial conditions of Eq. (2) in Eq. (18), in conjunction with Eq. (17), to derive

$$\alpha_0 = -\frac{A}{2} + a\,\frac{1}{\omega}\frac{1}{2i}\int_{-\infty}^{0} dt' F(t')e^{-i\omega t'} - b\,\frac{1}{\omega}\frac{1}{2i}\int_{0}^{+\infty} dt' F(t')e^{-i\omega t'}, \tag{19a}$$

$$\beta_0 = -\frac{A}{2} - c\,\frac{1}{\omega}\frac{1}{2i}\int_{-\infty}^{0} dt' F(t')e^{+i\omega t'} + d\,\frac{1}{\omega}\frac{1}{2i}\int_{0}^{+\infty} dt' F(t')e^{+i\omega t'}. \tag{19b}$$

Using the above expressions for α_0 and β_0 in Eq. (18) obtain

$$x(t) = -A\cos\omega t - \frac{1}{\omega}\int_0^t dt' F(t')\sin\omega(t-t'), \tag{20}$$

which uses $a+b=1$ and $c+d=1$.