Homework No. 04 (Fall 2013)

PHYS 520A: Electromagnetic Theory I

Due date: Thursday, 2013 Oct 17, 4.30pm

1. Conducting electrons, unlike bound electrons, are not confined to a particular atom. In the Drude model the conduction electrons are described by Newton's law

$$m\frac{d}{dt}\mathbf{v}(t) = -m\gamma\mathbf{v}(t) + e\mathbf{E}(t), \tag{1}$$

where the effects of collisions are modeled by a frictional force proportional (and opposite) to the velocity. If n_f is the (constant) density of (free) conduction electrons, then the conduction current density is given by

$$\mathbf{J}(t) = n_f e\mathbf{v}(t). \tag{2}$$

   (a) Solve the above differential equation and express the solution in the form

$$\mathbf{v}(t) = \frac{e}{m}\int_{-\infty}^{t} dt'\, e^{-\gamma(t-t')}\mathbf{E}(t'). \tag{3}$$

   (b) For a constant electric field show that the current density is given by

$$\mathbf{J} = \frac{n_f e^2}{m\gamma}\mathbf{E}. \tag{4}$$

   (c) Use the statement of Ohm's law,
$$\mathbf{J} = \sigma\mathbf{E}, \tag{5}$$

   to identify the expression for static conductivity σ.

   (d) Find the current density for

$$\mathbf{E}(t) = \mathbf{E}_0\frac{1}{\gamma_0}\delta(t), \tag{6}$$

   if $\mathbf{J}(t) = 0$ for $t < 0$. In particular, what is $\mathbf{J}$ immediately after $t = 0$?

2. A way of determining the sign of charge carriers in a conductor is by means of the Hall effect. A magnetic field $\mathbf{B}$ is applied perpendicular to the direction of current flow in a conductor, and as a consequence a transverse voltage drop appears across the conductor. If d is the transverse length of the conductor, and v is the average drift speed of the charge carriers, show that the voltage, in magnitude, is

$$V = vBd. \tag{7}$$

Estimate this potential drop (magnitude and direction) for a car driving towards North in the Northern hemisphere. How will the answer differ in the Southern hemisphere?

3. Use the statement of Ohm's law,
$$\mathbf{J} = \sigma\mathbf{E}, \tag{8}$$
and generalize it for a neutral conducting fluid moving with velocity $\mathbf{v}$ as
$$\mathbf{J} = \sigma(\mathbf{E} + \mathbf{v} \times \mathbf{B}). \tag{9}$$
Hint: In the co-moving coordinate system, in which the fluid is at rest, show that
$$\frac{\partial}{\partial t} \rightarrow \frac{d}{dt} = \frac{\partial}{\partial t} + \mathbf{v} \cdot \boldsymbol{\nabla}, \tag{10}$$
$$\mathbf{E} \rightarrow \mathbf{E} + \mathbf{v} \times \mathbf{B}. \tag{11}$$
Using Eq. (9) and Maxwell's equations derive
$$\frac{\partial \mathbf{B}}{\partial t} = \boldsymbol{\nabla} \times (\mathbf{v} \times \mathbf{B}) + \frac{c^2 \varepsilon_0}{\sigma}\nabla^2\mathbf{B}. \tag{12}$$
(Argue that the displacement current, ignored here, gives only v^2/c^2 corrections.) For a fluid at rest this means that $\mathbf{B}$ satisfies the diffusion equation
$$\frac{\partial \mathbf{B}}{\partial t} = \frac{1}{\mu_0 \sigma}\nabla^2\mathbf{B}. \tag{13}$$
If $\mathbf{B}$ varies over a characteristic distance L, what is the characteristic time τ for the decay of the field? Estimate τ for the Earth's core, where $L \sim 10^6\,\mathrm{m}$ and $\sigma \sim 10^7\,\mathrm{mho/m}$. Compare this time with the current estimates for geomagnetic reversal time.

4. Plot the following as a function of ω:

   (a) $\mathrm{Re}\chi(\omega)$ for an insulator ($\gamma \ll \omega_0$) in the Drude-Lorentz dielectric model.
   (b) $\mathrm{Im}\chi(\omega)$ for an insulator ($\gamma \ll \omega_0$) in the Drude-Lorentz dielectric model.
   (c) $\mathrm{Re}\chi(\omega)$ for a metal ($\omega_0 \ll \gamma$) in the Drude-Lorentz dielectric model.
   (d) $\mathrm{Im}\chi(\omega)$ for a metal ($\omega_0 \ll \gamma$) in the Drude-Lorentz dielectric model.

   Observe that the real part of the dielectric function (square of refractive index) represents dispersion. Anomalous dispersion is the behavior when the refractive index decreases with increase in frequency. Imaginary part of the dielectric function represents absorption. Is it correct to state that anomalous dispersion is accompanied by absorption?

5. Calculate the plasma frequency of gold using
$$\omega_p^2 = \frac{n_f e^2}{m\varepsilon_0}, \tag{14}$$
where n_f is the density of conduction electrons. Is this greater than or less than the frequency spectrum of visible light? Are good conductors always opaque and shiny to human eye?