Final Exam (Fall 2013)

PHYS 520A: Electromagnetic Theory I

Date: 2013 Dec 10

1. (**25 points.**) The electromagnetic energy density U and the corresponding energy flux vector $\mathbf{S}$ are given by, ($\mathbf{D} = \varepsilon_0 \mathbf{E}$, $\mathbf{B} = \mu_0 \mathbf{H}$, $\varepsilon_0 \mu_0 c^2 = 1$,)

$$U = \frac{1}{2}(\mathbf{D}\cdot\mathbf{E} + \mathbf{B}\cdot\mathbf{H}), \qquad \mathbf{S} = \mathbf{E}\times\mathbf{H}. \tag{1}$$

Th electromagnetic momentum density $\mathbf{G}$ and the corresponding momentum flux tensor $\mathbf{T}$ are given by

$$\mathbf{G} = \mathbf{D}\times\mathbf{B}, \qquad \mathbf{T} = \frac{1}{2}\mathbf{1}(\mathbf{D}\cdot\mathbf{E} + \mathbf{B}\cdot\mathbf{H}) - (\mathbf{DE} + \mathbf{BH}). \tag{2}$$

Show that

$$\mathrm{Tr}(\mathbf{T}) = T_{ii} = U \tag{3}$$

and

$$\mathrm{Tr}(\mathbf{T}\cdot\mathbf{T}) = T_{ij}T_{ji} = 3U^2 - 2\mathbf{G}\cdot\mathbf{S}. \tag{4}$$

2. (**25 points.**) A uniformly polarized sphere of radius R is described by, $n \neq -2$,

$$\mathbf{P}(\mathbf{r}) = \alpha r^n\, \hat{\mathbf{r}}\, \theta(R - r). \tag{5}$$

Find the effective charge density by calculating $-\boldsymbol{\nabla}\cdot\mathbf{P}$. In particular, you should obtain two terms, one containing $\theta(R-r)$ that is interpreted as a volume charge density, and another containing $\delta(R-r)$ that can be interpreted as a surface charge density.

3. (**25 points.**) The electrostatic electric potential, $\phi(\mathbf{r})$, for a unit point charge placed at the origin satisfies

$$-\nabla^2\phi(\mathbf{r}) = \delta^{(3)}(\mathbf{r}). \tag{6}$$

Verify, by substituting into Eq. (6), that

$$\phi(\mathbf{r}) = \frac{1}{4\pi r} \tag{7}$$

is a particular solution for $\phi(\mathbf{r})$.
Hint: Verify that the left hand side of Eq. (6) satisfies the properties of δ-function in three dimensions, i.e., it is zero for $\mathbf{r} \neq 0$ and the integral over a volume including $\mathbf{r} = 0$ is 1.

4. (**25 points.**) The modified Bessel functions, $I_m(k\rho)$ and $K_m(k\rho)$, satisfy the differential equation

$$\left[-\frac{1}{\rho}\frac{d}{d\rho}\rho\frac{d}{d\rho}+\frac{m^2}{\rho^2}+k^2\right]\left\{\begin{array}{c} I_m(k\rho) \\ K_m(k\rho) \end{array}\right\}=0. \tag{8}$$

Derive the identity, for the Wronskian, (upto a constant C)

$$I_m(k\rho)K_m'(k\rho)-K_m(k\rho)I_m'(k\rho)=-\frac{C}{k\rho}, \tag{9}$$

where

$$I_m'(t)\equiv\frac{d}{dt}I_m(t) \qquad \text{and} \qquad K_m'(t)\equiv\frac{d}{dt}K_m(t). \tag{10}$$

Further, determine the value of the constant C on the right hand side of Eq. (9) using the asymptotic forms for the modified Bessel functions:

$$I_m(t) \xrightarrow{t\gg 1} \frac{1}{\sqrt{2\pi}}\frac{e^t}{\sqrt{t}}, \tag{11}$$

$$K_m(t) \xrightarrow{t\gg 1} \sqrt{\frac{\pi}{2}}\frac{e^{-t}}{\sqrt{t}}. \tag{12}$$