Solutions

Final Exam (Fall 2013)

PHYS 520A: Electromagnetic Theory I

Date: 2013 Dec 10

1. **(25 points.)** The electromagnetic energy density U and the corresponding energy flux vector $\mathbf{S}$ are given by, ($\mathbf{D} = \varepsilon_0 \mathbf{E}$, $\mathbf{B} = \mu_0 \mathbf{H}$, $\varepsilon_0 \mu_0 c^2 = 1$,)

$$U = \frac{1}{2}(\mathbf{D}\cdot\mathbf{E} + \mathbf{B}\cdot\mathbf{H}), \qquad \mathbf{S} = \mathbf{E}\times\mathbf{H}. \tag{1}$$

Th electromagnetic momentum density $\mathbf{G}$ and the corresponding momentum flux tensor $\mathbf{T}$ are given by

$$\mathbf{G} = \mathbf{D}\times\mathbf{B}, \qquad \mathbf{T} = \frac{1}{2}\mathbf{1}(\mathbf{D}\cdot\mathbf{E} + \mathbf{B}\cdot\mathbf{H}) - (\mathbf{DE} + \mathbf{BH}). \tag{2}$$

Show that

$$\mathrm{Tr}(\mathbf{T}) = T_{ii} = U \tag{3}$$

and

$$\mathrm{Tr}(\mathbf{T}\cdot\mathbf{T}) = T_{ij}T_{ji} = 3U^2 - 2\mathbf{G}\cdot\mathbf{S}. \tag{4}$$

2. **(25 points.)** A uniformly polarized sphere of radius R is described by, $n \neq -2$,

$$\mathbf{P}(\mathbf{r}) = \alpha r^n \,\hat{\mathbf{r}}\,\theta(R - r). \tag{5}$$

Find the effective charge density by calculating $-\boldsymbol{\nabla}\cdot\mathbf{P}$. In particular, you should obtain two terms, one containing $\theta(R-r)$ that is interpreted as a volume charge density, and another containing $\delta(R-r)$ that can be interpreted as a surface charge density.

3. **(25 points.)** The electrostatic electric potential, $\phi(\mathbf{r})$, for a unit point charge placed at the origin satisfies

$$-\nabla^2\phi(\mathbf{r}) = \delta^{(3)}(\mathbf{r}). \tag{6}$$

Verify, by substituting into Eq. (6), that

$$\phi(\mathbf{r}) = \frac{1}{4\pi r} \tag{7}$$

is a particular solution for $\phi(\mathbf{r})$.
Hint: Verify that the left hand side of Eq. (6) satisfies the properties of δ-function in three dimensions, i.e., it is zero for $\mathbf{r} \neq 0$ and the integral over a volume including $\mathbf{r} = 0$ is 1.

Prob 1, Final Exam

$$\overleftrightarrow{T} = \frac{1}{2}\overleftrightarrow{1}(\vec{D}\cdot\vec{E} + \vec{B}\cdot\vec{H}) - (\vec{D}\vec{E} + \vec{B}\vec{H})$$

$$Tr(\overleftrightarrow{T}) = \frac{1}{2}3(\vec{D}\cdot\vec{E} + \vec{B}\cdot\vec{H}) - (\vec{D}\cdot\vec{E} + \vec{B}\cdot\vec{H})$$

$$= \frac{1}{2}(\vec{D}\cdot\vec{E} + \vec{B}\cdot\vec{H})$$

$$= U.$$

$$\overleftrightarrow{T} = U\overleftrightarrow{1} - (\vec{D}\vec{E} + \vec{B}\vec{H})$$

$$\overleftrightarrow{T}\cdot\overleftrightarrow{T} = \left[U\overleftrightarrow{1} - (\vec{D}\vec{E} + \vec{B}\vec{H})\right]\cdot\left[U\overleftrightarrow{1} - (\vec{D}\vec{E} + \vec{B}\vec{H})\right]$$

$$= U^2\overleftrightarrow{1} - 2U(\vec{D}\vec{E} + \vec{B}\vec{H}) + (\vec{D}\vec{E} + \vec{B}\vec{H})\cdot(\vec{D}\vec{E} + \vec{B}\vec{H})$$

$$Tr(\overleftrightarrow{T}\cdot\overleftrightarrow{T}) = 3U^2 - (\vec{D}\cdot\vec{E} + \vec{B}\cdot\vec{H})^2 + \left[(\vec{D}\cdot\vec{E})^2 + 2(\vec{D}\cdot\vec{H})(\vec{E}\cdot\vec{B}) + (\vec{B}\cdot\vec{H})^2\right]$$

$$= 3U^2 - 2(\vec{D}\cdot\vec{E})(\vec{B}\cdot\vec{H}) + 2(\vec{D}\cdot\vec{H})(\vec{E}\cdot\vec{B})$$

$$\vec{G}\cdot\vec{S} = (\vec{D}\times\vec{B})\cdot(\vec{E}\times\vec{H})$$

$$= (\vec{D}\cdot\vec{E})(\vec{B}\cdot\vec{H}) - (\vec{D}\cdot\vec{H})(\vec{B}\cdot\vec{E})$$

Thus,

$$Tr(\overleftrightarrow{T}\cdot\overleftrightarrow{T}) = 3U^2 - 2\vec{G}\cdot\vec{S}$$

Prob 2, Final Exam

$$\vec{\nabla}\cdot\vec{r} = 3$$
$$\vec{\nabla} r = \hat{r}$$
$$-\frac{\partial}{\partial r}\theta(R-r) = \delta(R-r)$$

$$-\vec{\nabla}\cdot\vec{P} = -\vec{\nabla}\cdot\left[\alpha r^n \hat{r}\, \theta(R-r)\right]$$
$$= -\alpha\, \vec{\nabla}\cdot\left[r^{n-1}\vec{r}\, \theta(R-r)\right]$$
$$= -\alpha\left(\vec{\nabla} r^{n-1}\right)\cdot\vec{r}\,\theta(R-r) - \alpha r^{n-1}(\vec{\nabla}\cdot\vec{r})\,\theta(R-r)$$
$$-\alpha r^{n-1}\vec{r}\cdot\vec{\nabla}\theta(R-r)$$
$$= -\alpha(n-1) r^{n-2}(\vec{\nabla} r)\cdot\vec{r}\,\theta(R-r) - 3\alpha r^{n-1}\theta(R-r)$$
$$-\alpha r^{n-1}\vec{r}\cdot(\vec{\nabla} r)\frac{\partial}{\partial r}\theta(R-r)$$
$$= -\alpha(n-1) r^{n-1}\theta(R-r) - 3\alpha r^{n-1}\theta(R-r) + \alpha r^n \delta(r-R)$$
$$= -(n+2)\,\alpha r^{n-1}\theta(R-r) + \alpha r^n\delta(r-R)$$

Prob 3, Final Exam

$$-\nabla^2\phi = -\vec{\nabla}\cdot\vec{\nabla}\frac{1}{4\pi r}$$
$$= \vec{\nabla}\cdot\left[\frac{\vec{r}}{4\pi r^3}\right]$$
$$= \frac{\vec{\nabla}\cdot\vec{r}}{4\pi r^3} + \frac{\vec{r}}{4\pi}\cdot\vec{\nabla}\frac{1}{r^3}$$
$$= \frac{3}{4\pi r^3} + \frac{\vec{r}}{4\pi}\cdot\frac{(-3)}{r^4}\hat{r} = 0 \quad ✓$$

$$-\int d^3r\, \nabla^2\phi = -\oint d\vec{s}\cdot\vec{\nabla}\frac{1}{4\pi r}$$
$$= \oint d\vec{s}\cdot\frac{\vec{r}}{4\pi r^3} = 4\pi r^2\hat{r}\cdot\frac{\vec{r}}{4\pi r^3} = 1. \quad ✓$$

Prob 4, Final Exam

$$K_m(k\rho)\left[-\frac{1}{\rho}\frac{d}{d\rho}\rho\frac{d}{d\rho} + \frac{m^2}{\rho^2} + k^2\right] I_m(k\rho) = 0$$

$$I_m(k\rho)\left[-\frac{1}{\rho}\frac{d}{d\rho}\rho\frac{d}{d\rho} + \frac{m^2}{\rho^2} + k^2\right] K_m(k\rho) = 0$$

Subtracting the two equations we have.

$$I_m(k\rho)\frac{d}{d\rho}\rho\frac{d}{d\rho}K_m(k\rho) - K_m(k\rho)\frac{d}{d\rho}\rho\frac{d}{d\rho}I_m(k\rho) = 0$$

$$\Rightarrow \frac{d}{d\rho}\left[\rho\, I_m(k\rho)\frac{d}{d\rho}K_m(k\rho) - \rho\, K_m(k\rho)\frac{d}{d\rho}I_m(k\rho)\right] = 0$$

$$\Rightarrow I_m(k\rho)\frac{d}{d\rho}K_m(k\rho) - K_m(k\rho)\frac{d}{d\rho}I_m(k\rho) = -\frac{c}{\rho}$$

For $k\rho \gg 1$

$$-\frac{c}{\rho} = \frac{e^{k\rho}}{\sqrt{2\pi k\rho}}\frac{d}{d\rho}\left(\sqrt{\frac{\pi}{2}}\frac{e^{-k\rho}}{\sqrt{k\rho}}\right) - \sqrt{\frac{\pi}{2}}\frac{e^{-k\rho}}{\sqrt{k\rho}}\frac{d}{d\rho}\frac{e^{k\rho}}{\sqrt{2\pi k\rho}}$$

$$= -\frac{k}{2k\rho} - \frac{k}{2k\rho} = -\frac{1}{\rho}$$

$$\Rightarrow \quad c = 1.$$