(Bonus Take-Home) Exam No. 04 (Fall 2013)

PHYS 520A: Electromagnetic Theory I

Due date: Wednesday, 2013 Nov 6, 4.30pm

1. (Based on Griffiths 4th ed., Problem 4.10.) Consider a uniformly polarized sphere of radius R described by

$$\mathbf{P}(\mathbf{r}) = \alpha\,\mathbf{r}\,\theta(R-r). \tag{1}$$

(a) Calculate $-\boldsymbol{\nabla}\cdot\mathbf{P}$. Thus, find the effective charge density to be

$$\rho_{\text{eff}} = -3\alpha\theta(R-r) + \alpha r\delta(r-R). \tag{2}$$

(b) Using

$$\phi(\mathbf{r}) = \frac{1}{4\pi\varepsilon_0}\int d^3r'\,\frac{\rho_{\text{eff}}(\mathbf{r}')}{|\mathbf{r}-\mathbf{r}'|}, \tag{3}$$

evaluate the electric potential to be

$$\phi(\mathbf{r}) = \begin{cases} -\dfrac{\alpha}{2\varepsilon_0}(R^2-r^2), & r<R, \\ 0, & R<r. \end{cases} \tag{4}$$

(Hint: Choose $\mathbf{r}$ along $\hat{\mathbf{z}}$.)

(c) Evaulate the electric field

$$\mathbf{E}(\mathbf{r}) = -\boldsymbol{\nabla}\phi(\mathbf{r}) = \begin{cases} -\dfrac{\alpha}{\varepsilon_0}\,\mathbf{r}, & r<R, \\ 0, & r>R. \end{cases} \tag{5}$$

(d) Find the enclosed charge inside a sphere of radius r using

$$Q_{\text{en}} = \int d^3r'\,\rho_{\text{eff}}(\mathbf{r}') \tag{6}$$

for $r<R$ and $r>R$.

(e) Use Gauss's law,

$$\oint d\mathbf{a}\cdot\mathbf{E} = \frac{1}{\varepsilon_0}\,Q_{\text{en}}, \tag{7}$$

to verify the expression for the electric field in Eq. (5).

(f) Interpret the electric field for $r>R$ as the electric field due to the total charge inside $r\le R$.