Solutions

(Bonus Take-Home) Exam No. 04 (Fall 2013)

PHYS 520A: Electromagnetic Theory I

Due date: Wednesday, 2013 Nov 6, 4.30pm

1. (Based on Griffiths 4th ed., Problem 4.10.) Consider a uniformly polarized sphere of radius R described by

$$\mathbf{P}(\mathbf{r}) = \alpha\,\mathbf{r}\,\theta(R-r). \tag{1}$$

(a) Calculate $-\boldsymbol{\nabla}\cdot\mathbf{P}$. Thus, find the effective charge density to be

$$\rho_{\text{eff}} = -3\alpha\theta(R-r) + \alpha r\delta(r-R). \tag{2}$$

(b) Using

$$\phi(\mathbf{r}) = \frac{1}{4\pi\varepsilon_0}\int d^3r'\,\frac{\rho_{\text{eff}}(\mathbf{r}')}{|\mathbf{r}-\mathbf{r}'|}, \tag{3}$$

evaluate the electric potential to be

$$\phi(\mathbf{r}) = \begin{cases} -\dfrac{\alpha}{2\varepsilon_0}(R^2-r^2), & r<R,\\ 0, & R<r. \end{cases} \tag{4}$$

(Hint: Choose $\mathbf{r}$ along $\hat{\mathbf{z}}$.)

(c) Evaulate the electric field

$$\mathbf{E}(\mathbf{r}) = -\boldsymbol{\nabla}\phi(\mathbf{r}) = \begin{cases} -\dfrac{\alpha}{\varepsilon_0}\mathbf{r}, & r<R,\\ 0, & r>R. \end{cases} \tag{5}$$

(d) Find the enclosed charge inside a sphere of radius r using

$$Q_{\text{en}} = \int d^3r'\,\rho_{\text{eff}}(\mathbf{r}') \tag{6}$$

for $r<R$ and $r>R$.

(e) Use Gauss's law,

$$\oint d\mathbf{a}\cdot\mathbf{E} = \frac{1}{\varepsilon_0}Q_{\text{en}}, \tag{7}$$

to verify the expression for the electric field in Eq. (5).

(f) Interpret the electric field for $r>R$ as the electric field due to the total charge inside $r\le R$.

(Bonus Take-Home) Exam No. 4

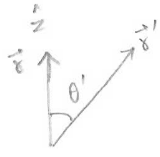
We have a polarized sphere of radius R,

$$\vec{P}(\vec{r}) = \alpha\, \vec{r}\, \theta(R-r)$$

(a)
$$\begin{aligned}
\rho_{eff} &= -\vec{\nabla}\cdot\vec{P} \\
&= -\vec{\nabla}\cdot\left[\alpha\, \vec{r}\, \theta(R-r)\right] \\
&= -\alpha\,(\vec{\nabla}\cdot\vec{r})\,\theta(R-r) - \alpha\, \vec{r}\cdot\vec{\nabla}\,\theta(R-r) \\
&= -3\alpha\,\theta(R-r) - \alpha\, \vec{r}\cdot\hat{r}\,\frac{\partial}{\partial r}\theta(R-r) \\
&= -3\alpha\,\theta(R-r) + \alpha r\,\delta(R-r).
\end{aligned}$$

(b)
$$\phi(\vec{r}) = \frac{1}{4\pi\epsilon_0}\int d^3r'\,\frac{\rho_{eff}(r')}{|\vec{r}-\vec{r}'|}$$

$$= \frac{1}{4\pi\epsilon_0}\int d^3r'\,\frac{\left[-3\alpha\,\theta(R-r') + \alpha r'\delta(R-r')\right]}{\sqrt{r^2+r'^2-2\vec{r}\cdot\vec{r}'}}$$



$$= \frac{1}{4\pi\epsilon_0}\int_0^{2\pi} d\phi'\int_0^{\pi}\sin\theta' d\theta'\int_0^{\infty} r'^2 dr'\,\frac{\left[-3\alpha\,\theta(R-r') + \alpha r'\,\delta(R-r')\right]}{\sqrt{r^2+r'^2-2rr'\cos\theta'}}$$

$$= \frac{2\pi}{4\pi\epsilon_0}\int_0^{\infty} r'^2 dr'\left[-3\alpha\,\theta(R-r') + \alpha r'\,\delta(R-r')\right]\int_0^{\pi}\frac{\sin\theta' d\theta'}{\sqrt{r^2+r'^2-2rr'\cos\theta'}}$$

$$I(r, r') = \int_0^{\pi} \sin\theta' d\theta' \frac{1}{\sqrt{r^2 + r'^2 - 2rr'\cos\theta'}} \qquad \cos\theta' = t$$

$$= \int_{-1}^{1} dt \frac{1}{\sqrt{r^2 + r'^2 - 2rr't}} \qquad \begin{array}{r} r^2 + r'^2 - 2rr't = y \\ -2rr'dt = dy \end{array}$$

$$= \int_{(r+r')^2}^{(r-r')^2} \frac{-dy}{2rr'} \frac{1}{\sqrt{y}}$$

$$= \frac{1}{2rr'} \int_{(r-r')^2}^{(r+r')^2} \frac{dy}{\sqrt{y}}$$

$$= \frac{1}{rr'} \sqrt{y} \Big|_{(r-r')^2}^{(r+r')^2}$$

$$= \frac{1}{rr'} \left[(r+r') - |r-r'| \right]$$

$$= \frac{1}{rr'} 2 \, \mathrm{Min}(r, r')$$

$$= \frac{2}{\mathrm{Max}(r, r')}$$

Using $I(r, r')$ in the expression for $\phi(\vec{r})$ we have.

$$\phi(\vec{r}) = \frac{1}{2\epsilon_0} \int_0^{\infty} r'^2 dr' \left[-3\alpha\,\theta(R-r') + \alpha r' \delta(R-r') \right] \frac{2}{\mathrm{Max}(r, r')}$$

$$= -\frac{3\alpha}{\epsilon_0} \int_0^{R} dr' \frac{r'^2}{\mathrm{Max}(r, r')} + \frac{\alpha}{\epsilon_0} \frac{R^3}{\mathrm{Max}(r, R)}$$

$$\int_0^R dr' \frac{r'^2}{\mathrm{Max}(r,r')} = \begin{cases} \displaystyle\int_0^r dr' \frac{r'^2}{r} + \int_r^R dr' \frac{r'^2}{r'} & r < R \\ \displaystyle\int_0^R dr' \frac{r'^2}{r} & R < r \end{cases}$$

$$= \begin{cases} \dfrac{r^2}{3} + \dfrac{R^2}{2} - \dfrac{r^2}{2} & r < R \\ \dfrac{R^3}{3r} & R < r \end{cases}$$

$$= \begin{cases} \dfrac{R^2}{2} - \dfrac{r^2}{6} & r < R \\ \dfrac{R^3}{3r} & R < r \end{cases}$$

Thus, we have.

$$\phi(\vec{r}) = \begin{cases} -\dfrac{3\alpha}{\epsilon_0}\left(\dfrac{R^2}{2} - \dfrac{r^2}{6}\right) + \dfrac{\alpha}{\epsilon_0} R^2 & r < R \\ -\dfrac{3\alpha}{\epsilon_0}\dfrac{R^3}{3r} + \dfrac{\alpha}{\epsilon_0}\dfrac{R^3}{r} & R < r \end{cases}$$

$$= \begin{cases} -\dfrac{\alpha}{2\epsilon_0}\left(R^2 - r^2\right) & r < R \\ 0 & R < r. \end{cases}$$

(c) $\vec{E}(\vec{r}) = -\vec{\nabla}\phi(\vec{r})$

$$= \begin{cases} +\dfrac{\alpha}{2\epsilon_0}\vec{\nabla}(R^2 - r^2) & r < R \\ 0 & R < r \end{cases}$$

$$= \begin{cases} -\dfrac{\alpha}{\epsilon_0}\vec{r} & r < R \\ 0 & R < r. \end{cases}$$

(d) Charge enclosed inside a sphere of radius r

$$Q_{en} = \int_V d^3r' \, \rho_{eff}(\vec{r}')$$

$$= \int_V d^3r' \left[-3\alpha\, \theta(R - r') + \alpha r' \delta(r' - R)\right]$$

$$= 4\pi \int_0^r r'^2 dr' \left[-3\alpha\, \theta(R - r') + \alpha r' \delta(r' - R)\right]$$

$$= -12\pi\alpha \int_0^{\mathrm{Min}(r,R)} r'^2 dr' + 4\pi\alpha \int_0^r r'^3 dr' \, \delta(r' - R)$$

$$= \begin{cases} -12\pi\alpha \dfrac{r^3}{3} + 0 & r < R \\ -12\pi\alpha \dfrac{R^3}{3} + 4\pi\alpha R^3 & R < r \end{cases}$$

$$= \begin{cases} -4\pi\alpha r^3 & r < R \\ 0 & R < r \end{cases}$$

(e) $$\oint d\vec{a} \cdot \vec{E} = \frac{1}{\epsilon_0} Q_{en}.$$

$$4\pi r^2 E = \begin{cases} -\frac{4\pi\alpha}{\epsilon_0} r^3 & r < R \\ 0 & R < r \end{cases}$$

$$E = \begin{cases} -\frac{\alpha}{\epsilon_0} r & r < R \\ 0 & R < r \end{cases}$$

$$\vec{E} = \begin{cases} -\frac{\alpha}{\epsilon_0} \vec{r} & r < R \\ 0 & R < r. \end{cases}$$ ✓

(f) For a spherically symmetric neutral charge distribution the electric field is indeed zero outside the charge distribution.