Solutions

Exam No. 03 (Fall 2013)

PHYS 520A: Electromagnetic Theory I

Date: 2013 Nov 21

1. (**30 points.**) A uniformly polarized sphere of radius R is described by

$$\mathbf{P}(\mathbf{r}) = \alpha r^2 \,\hat{\mathbf{r}}\, \theta(R - r). \tag{1}$$

Find the effective charge density by calculating $-\boldsymbol{\nabla} \cdot \mathbf{P}$. In particular, you should obtain two terms, one containing $\theta(R-r)$ that is interpreted as a volume charge density, and another containing $\delta(R-r)$ that can be interpreted as a surface charge density.

2. (**30 points.**) Consider the Green's function equation

$$-\left(\frac{d^2}{dt^2} + \omega^2\right) G(t) = \delta(t). \tag{2}$$

Verify, by substituting into Eq. (2), that

$$G(t) = -\frac{1}{\omega}\theta(t) \sin \omega(t), \tag{3}$$

is a particular solution to the Green's function equation.

3. (**40 points.**) The expression for the electric potential due to a point charge placed in between two perfectly conducting semi-infinite slabs described by

$$\varepsilon(z) = \begin{cases} \infty, & z < 0, \\ \varepsilon_0, & 0 < z < a, \\ \infty, & a < z, \end{cases} \tag{4}$$

is given in terms of the reduced Green's function that satisfies the differential equation $(0 < \{z, z'\} < a)$

$$\left[-\frac{\partial^2}{\partial z^2} + k^2\right] \varepsilon_0 g(z, z') = \delta(z - z') \tag{5}$$

with boundary conditions requiring the reduced Green's function to vanish at $z = 0$ and $z = a$.

(a) Construct the reduced Green's function in the form

$$g(z, z') = \begin{cases} A \sinh kz + B \cosh kz, & 0 < z < z' < a, \\ C \sinh kz + D \cosh kz, & 0 < z' < z < a, \end{cases} \tag{6}$$

and solve for the four coefficients, A, B, C, D, using the conditions

$$g(0, z') = 0, \tag{7a}$$

$$g(a, z') = 0, \tag{7b}$$

$$g(z, z')\Big|_{z=z'-\delta}^{z=z'+\delta} = 0, \tag{7c}$$

$$\varepsilon_0 \partial_z g(z, z')\Big|_{z=z'-\delta}^{z=z'+\delta} = -1. \tag{7d}$$

Hint: The hyperbolic functions here are defined as

$$\sinh x = \frac{1}{2}(e^x - e^{-x}) \qquad \text{and} \qquad \cosh x = \frac{1}{2}(e^x + e^{-x}). \tag{8}$$

(b) Take the limit $ka \to \infty$ in your solution above to obtain the reduced Green's function for a single perfectly conducting slab,

$$\lim_{ka\to\infty} g(z, z') = \frac{1}{\varepsilon_0}\frac{1}{2k} e^{-k|z-z'|} - \frac{1}{\varepsilon_0}\frac{1}{2k} e^{-k|z|} e^{-k|z'|}. \tag{9}$$

This should serve as a check for your solution to the reduced Green's function.

Prob 1, Exam No. 3

$$\vec{P} = \alpha r^2 \hat{r}\, \theta(R-r)$$

$$= \alpha r\, \vec{r}\, \theta(R-r)$$

$$-\vec{\nabla}\cdot\vec{P} = -\vec{\nabla}\cdot\left[\alpha r \vec{r}\, \theta(R-r)\right]$$

$$= -\alpha(\vec{\nabla} r)\cdot\vec{r}\,\theta(R-r) - \alpha r\, \vec{\nabla}\cdot\vec{r}\,\theta(R-r) - \alpha r \vec{r}\cdot\hat{r}\frac{\partial}{\partial r}\theta(R-r)$$

$$= -\alpha\, \hat{r}\cdot\vec{r}\,\theta(R-r) - 3\alpha r\,\theta(R-r) + \alpha r^2 \delta(R-r)$$

$$= -\alpha r\,\theta(R-r) - 3\alpha r\,\theta(R-r) + \alpha r^2\delta(R-r)$$

$$= -4\alpha r\,\theta(R-r) + \alpha r^2 \delta(R-r)$$

Prob 2, Exam No. 3

$$G(t) = -\frac{1}{\omega}\theta(t)\, \sin\omega t$$

$$\frac{d}{dt}G(t) = -\frac{1}{\omega}\delta(t)\sin\omega t - \theta(t)\cos\omega t$$

$$\frac{d^2}{dt^2}G(t) = -\frac{1}{\omega}\left(\frac{d}{dt}\delta(t)\right)\sin\omega t - 2\delta(t)\cos\omega t + \omega\,\theta(t)\sin\omega t$$

$$= -t\left(\frac{d}{dt}\delta(t)\right)\frac{\sin\omega t}{\omega t} - 2\delta(t)\cos\omega t - \omega^2 G(t)$$

$$-\left(\frac{d^2}{dt^2}+\omega^2\right)G(t) = 2\delta(t) + t\left(\frac{d}{dt}\delta(t)\right)\frac{\sin\omega t}{\omega t}.$$

Note that $\bar{\delta}(t) = -t \frac{d}{dt} \delta(t)$ is a model for delta function.

$$\int_{-\infty}^{+\infty} dt \; \bar{\delta}(t) = - \int_{-\infty}^{+\infty} dt \; t \frac{d}{dt} \delta(t)$$

$$= - \int_{-\infty}^{+\infty} dt \frac{d}{dt} \left(t \; \delta(t) \right) + \int_{-\infty}^{+\infty} dt \; \delta(t)$$

$$= 0 + 1 = 1 .$$

Thus, we have.

$$-\left(\frac{d^2}{dt^2} + \omega^2\right) G(t) = 2\,\delta(t) - \delta(t) \frac{\sin \omega t}{\omega t}$$

$$\underset{\omega t \to 0}{Lt} \frac{\sin \omega t}{\omega t} = 1$$

$$= \delta(t)$$

Prob. 3, Exam - 3

$$\left[-\frac{\partial^2}{\partial z^2} + k_\perp^2\right] \epsilon_0\, g(z,z') = \delta(z-z') \qquad 0 < \{z,z'\} < a$$

with $\quad g(0,z') = 0 \quad$ and $\quad g(a,z') = 0 .$

(a) $$g(z,z') = \begin{cases} A \sinh k_\perp z + B \cosh k_\perp z & 0 < z < z' < a \\ C \sinh k_\perp z + D \cosh k_\perp z & 0 < z' < z < a. \end{cases}$$

$$g(0,z') = 0 \Rightarrow A \sinh 0 + B \cosh 0 = 0$$

$$\Rightarrow B = 0 .$$

$$g(a,z') = 0 \Rightarrow C \sinh k_\perp a + D \cosh k_\perp a = 0.$$

$$C = -D \frac{\cosh k_\perp a}{\sinh k_\perp a}$$

Thus,

$$C \sinh k_\perp z + D \cosh k_\perp z = -D \frac{\cosh k_\perp a}{\sinh k_\perp a} \sinh k_\perp z + D \cosh k_\perp z$$

$$= \frac{D}{\sinh k_\perp a}\left[\sinh k_\perp a \cosh k_\perp z - \cosh k_\perp a \sinh k_\perp z\right]$$

$$= \frac{D}{\sinh k_\perp a} \sinh k_\perp (a-z)$$

$$g(z,z') = \begin{cases} A \sinh k_1 z & 0 < z < z' < a. \\ D \dfrac{\sinh k_1 (a-z)}{\sinh k_1 a} & 0 < z' < z < a. \end{cases}$$

Continuity conditions

$$g \Big|_{z=z'-\delta}^{z=z'+\delta} = 0$$

$$\frac{\partial}{\partial z} g \Big|_{z=z'-\delta}^{z=z'+\delta} = -\frac{1}{\epsilon_0}$$

leads to

$$D \frac{\sinh k_1 (a-z')}{\sinh k_1 a} - A \sinh k_1 z' = 0$$

$$+ D \frac{\cosh k_1 (a-z')}{\sinh k_1 a} + A \cosh k_1 z' = + \frac{1}{k_1 \epsilon_0}$$

$$\text{Denominator} = \frac{1}{\sinh k_1 a} \left[\sinh k_1 (a-z') \cosh k_1 z' + \cosh k_1 (a-z') \sinh k_1 z' \right]$$

$$= 1 .$$

$$D = \frac{1}{k_1 \epsilon_0} \sinh k_1 z'$$

$$A = \frac{1}{k_1 \epsilon_0} \frac{\sinh k_1 (a-z')}{\sinh k_1 a} .$$

$$g(z,z') = \begin{cases} \dfrac{1}{k_1\epsilon_0} \dfrac{\sinh k_1(a-z') \sinh k_1 z}{\sinh k_1 a} & 0 < z < z' < a \\[2ex] \dfrac{1}{k_1\epsilon_0} \dfrac{\sinh k_1(a-z) \sinh k_1 z'}{\sinh k_1 a} & 0 < z' < z < a \end{cases}$$

$$= \frac{\sinh k_1(a - z_>) \; \sinh k_1 z_<}{k_1\epsilon_0 \; \sinh k_1 a}$$

$z_< = \mathrm{Min}(z,z')$
$z_> = \mathrm{Max}(z,z')$

(b)

$$g(z,z') = \frac{1}{2k_1\epsilon_0} \frac{\left(e^{k_1 a - k_1 z_>} - e^{k_1 z_> - k_1 a}\right)\left(e^{k_1 z_<} - e^{-k_1 z_<}\right)}{\left(e^{k_1 a} - e^{-k_1 a}\right)}$$

$$= \frac{1}{\epsilon_0}\frac{1}{2k_1} \frac{\left(e^{-k_1 z_>} - e^{k_1 z_>} e^{-2k_1 a}\right)}{1 - e^{-2k_1 a}} \left(e^{k_1 z_<} - e^{-k_1 z_<}\right)$$

$$= \frac{1}{\epsilon_0}\frac{1}{2k_1} e^{-k_1 z_>}\left(e^{k_1 z_<} - e^{-k_1 z_<}\right)$$

$$= \frac{1}{\epsilon_0}\frac{1}{2k_1} e^{-k_1(z_> - z_<)} - \frac{1}{\epsilon_0}\frac{1}{2k_1} e^{-k_1 z_>} e^{-k_1 z_<}$$

$$= \frac{1}{\epsilon_0}\frac{1}{2k_1} e^{-k_1|z-z'|} - \frac{1}{\epsilon_0}\frac{1}{2k_1} e^{-k_1|z|} e^{-k_1|z'|}$$

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