Solutions

Exam No. 02 (Fall 2013)

PHYS 520A: Electromagnetic Theory I

Date: 2013 Oct 24

1. Show that the effective charge density, ρ_{eff}, and the effective current density, $\mathbf{j}_{\text{eff}}$,

$$\rho_{\text{eff}} = -\boldsymbol{\nabla} \cdot \mathbf{P}, \tag{1}$$

$$\mathbf{j}_{\text{eff}} = \frac{\partial}{\partial t}\mathbf{P} + \boldsymbol{\nabla} \times \mathbf{M}, \tag{2}$$

satisfy the equation of charge conservation

$$\frac{\partial}{\partial t}\rho_{\text{eff}} + \boldsymbol{\nabla} \cdot \mathbf{j}_{\text{eff}} = 0. \tag{3}$$

2. Consider the charge density

$$\rho(\mathbf{r}) = -\mathbf{d} \cdot \boldsymbol{\nabla}\delta^{(3)}(\mathbf{r}). \tag{4}$$

(a) Find the total charge of the charge density by evaluating

$$\int d^3r\, \rho(\mathbf{r}). \tag{5}$$

(b) Find the dipole moment of the charge density by evaluating

$$\int d^3r\, \mathbf{r}\, \rho(\mathbf{r}). \tag{6}$$

3. The response to an electric field in the Drude model is described by the susceptibility function

$$\chi(\omega) = \frac{\omega_p^2}{\omega_0^2 - i\omega\gamma}. \tag{7}$$

Plot $[\text{Re}\chi(\omega)]$ as a function of ω.

4. Consider a circular loop of wire carrying current I whose magnetic moment is given by $\boldsymbol{\mu} = IA\hat{\mathbf{n}}$, where $\hat{\mathbf{n}}$ points perpendicular to the plane containing the loop (satisfying the right hand sense) and A is the area of the loop. Consider the case $\hat{\mathbf{n}} = \hat{\mathbf{x}}$. What is the magnitude and direction of the torque experienced by this loop in the presence of a uniform magnetic field $\mathbf{B} = B\hat{\mathbf{y}}$. Describe the resultant motion of the loop. (Hint: The torque experienced by a magnetic moment $\boldsymbol{\mu}$ in a magnetic field $\mathbf{B}$ is $\boldsymbol{\tau} = \boldsymbol{\mu} \times \mathbf{B}$.)

5. A simple model for susceptibility is

$$\chi(\omega) = \frac{\omega_1}{\omega_0 - \omega} + i\,\pi\omega_1\delta(\omega - \omega_0), \tag{8}$$

where ω_0 and ω_1 represent physical parameters of a material.

(a) Note that

$$[\mathrm{Re}\chi(\omega)] = \frac{\omega_1}{\omega_0 - \omega} \qquad \text{and} \qquad [\mathrm{Im}\chi(\omega)] = \pi\omega_1\delta(\omega - \omega_0). \tag{9}$$

(b) Plot $[\mathrm{Re}\chi(\omega)]$ and $[\mathrm{Im}\chi(\omega)]$ with respect to ω.

(c) Evaluate the right hand side of the Kramers-Kronig relation

$$[\mathrm{Re}\chi(\omega)] = \lim_{\delta\to 0+} \int_{-\infty}^{\infty} \frac{d\omega'}{2\pi} [\mathrm{Im}\chi(\omega')]\, 2\mathrm{Re}\left\{\frac{1}{\omega' - (\omega + i\delta)}\right\} \tag{10}$$

for this simple model.

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Prob 1, Exam-2

$$\frac{\partial}{\partial t}\rho_{eff} + \vec{\nabla}\cdot\vec{j}_{eff} = \frac{\partial}{\partial t}(-\vec{\nabla}\cdot\vec{P}) + \vec{\nabla}\cdot\left[\frac{\partial}{\partial t}\vec{P} + \vec{\nabla}\times\vec{M}\right]$$

$$= 0.$$

Prob 2, Exam-2

$$\rho(\vec{r}) = -\vec{d}\cdot\vec{\nabla}\,\delta^{(3)}(\vec{r})$$

(a)
$$\int d^3r\,\rho(\vec{r}) = -\int d^3r\,\vec{d}\cdot\vec{\nabla}\,\delta^{(3)}(\vec{r})$$

$$= -\vec{d}\cdot\int d^3r\,\vec{\nabla}\,\delta^{(3)}(\vec{r})$$

$$= 0$$

(b)
$$\int d^3r\,\vec{r}\,\rho(\vec{r}) = -\int d^3r\,\vec{r}\,\vec{d}\cdot\vec{\nabla}\,\delta^{(3)}(\vec{r})$$

$$= -\int d^3r\,\vec{\nabla}\cdot\left[\vec{d}\,\vec{r}\,\delta^{(3)}(\vec{r})\right] + \int d^3r\left[\vec{\nabla}\cdot(\vec{d}\,\vec{r})\right]\delta^{(3)}(\vec{r})$$

$$= 0 + \int d^3r\left[\vec{d}\cdot(\vec{\nabla}\,\vec{r})\right]\delta^{(3)}(\vec{r})$$

$$= \int d^3r\,\vec{d}\cdot\overleftrightarrow{I}\,\delta^{(3)}(\vec{r})$$

$$= \vec{d}\int d^3r\,\delta^{(3)}(\vec{r})$$

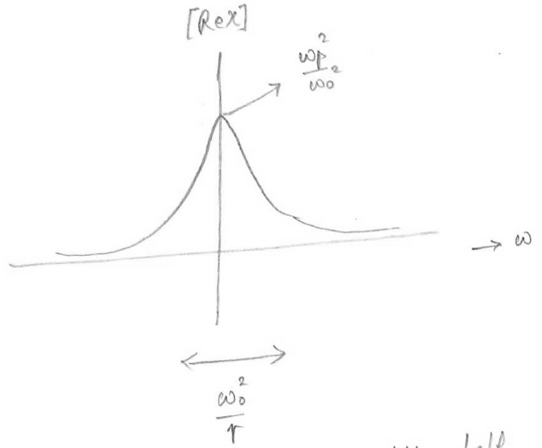
$$= \vec{d}.$$

Prob 3, Exam-2

$$\chi(\omega) = \frac{\omega_p^2}{\omega_0^2 - i\omega\gamma}$$

$$[\mathrm{Re}\ \chi(\omega)] = \frac{\omega_p^2\,\omega_0^2}{\omega_0^4 + \omega^2\gamma^2}$$

Lorentz distribution.



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Prob 4, Exam-2

$$\vec{\mu} = I A\ \hat{x}$$

$$\vec{B} = B\ \hat{y}$$

$$\vec{\tau} = \vec{\mu} \times \vec{B} = I A B\ \hat{x} \times \hat{y}$$

$$= I A B\ \hat{z}$$

The loop will rotate about the z-axis, trying to align the direction of $\vec{\mu}$ with the direct of $\vec{B}$.

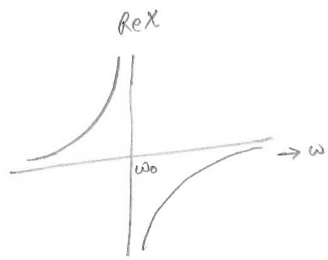
Prob 5, Exam-2

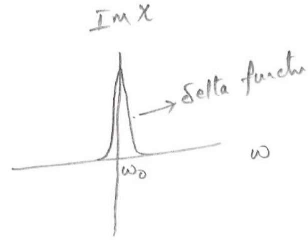
$$\chi(\omega) = \frac{\omega_1}{\omega_0 - \omega} + i\pi\omega_1 \delta(\omega - \omega_0)$$

(a) $$[\mathrm{Re}\,\chi(\omega)] = \frac{\omega_1}{\omega_0 - \omega}$$

$$[\mathrm{Im}\,\chi(\omega)] = \pi\omega_1 \delta(\omega - \omega_0)$$

(b)





(c) $$\underset{\delta\to 0+}{\mathrm{Lt}} \int_{-\infty}^{+\infty} \frac{d\omega'}{2\pi} [\mathrm{Im}\,\chi(\omega')] \;\mathrm{Re}\, \frac{2}{\omega' - (\omega + i\delta)}$$

$$= \underset{\delta\to 0+}{\mathrm{Lt}} \int_{-\infty}^{+\infty} \frac{d\omega'}{\cancel{2\pi}} \cancel{\pi}\omega_1 \delta(\omega - \omega_0) \;\mathrm{Re}\, \frac{\cancel{2}}{\omega' - (\omega + i\delta)}$$

$$= \underset{\delta\to 0+}{\mathrm{Lt}} \;\mathrm{Re}\, \frac{\omega_1}{\omega_0 - (\omega + i\delta)}$$

$$= \frac{\omega_1}{\omega_0 - \omega}$$

$$= [\mathrm{Re}\,\chi(\omega)] \quad \checkmark$$