Solutions

Exam No. 01 (Fall 2013)

PHYS 520A: Electromagnetic Theory I

Date: 2013 Sep 19

1. Show that

$$\nabla(\hat{\mathbf{r}} \cdot \mathbf{a}) = -\frac{1}{r}\, \hat{\mathbf{r}} \times (\hat{\mathbf{r}} \times \mathbf{a}) \tag{1}$$

for a uniform (homogeneous in space) vector $\mathbf{a}$.

2. (Schwinger et al., problem 7, chapter 1.) A charge q moves in the vacuum under the influence of uniform fields $\mathbf{E}$ and $\mathbf{B}$. Assume that $\mathbf{E} \cdot \mathbf{B} = 0$ and $\mathbf{v} \cdot \mathbf{B} = 0$.

   (a) At what velocity does the charge move without acceleration?

   (b) What is the speed when $\varepsilon_0 E^2 = \mu_0 H^2$?

3. A plane wave is incident, in vacuum, on a perfectly absorbing flat screen.

   (a) Without compromising generality we can choose the screen at $z = z_a$. Starting with the statement of conservation of linear momentum,

$$\frac{\partial \mathbf{G}}{\partial t} + \nabla \cdot \mathbf{T} + \mathbf{f} = 0, \tag{2}$$

   integrate on the volume between $z = z_a - \delta$ and $z = z_a + \delta$ for infinitely small $\delta > 0$. Interpret the integral of force density $\mathbf{f}$ as the total force, $\mathbf{F}$, on the plate. Further, note that the integral of momentum density $\mathbf{G}$ goes to zero for infinitely small δ. Thus, obtain

$$\mathbf{F} = -\int_{-\infty}^{\infty} dx \int_{-\infty}^{\infty} dy \int_{z_a-\delta}^{z_a+\delta} dz\, \nabla \cdot \mathbf{T}. \tag{3}$$

   (b) Use divergence theorem to conclude

$$\mathbf{F} = -\oint d\mathbf{a} \cdot \mathbf{T}, \tag{4}$$

   where the closed surface encloses the volume between $z = z_a - \delta$ and $z = z_a + \delta$ for infinitely small $\delta > 0$. Choose the plane wave to be incident on the side $z = z_a - \delta$ of the plate, and assuming $\mathbf{E} = 0$ and $\mathbf{B} = 0$ on the side $z = z_a + \delta$, conclude that

$$\frac{\mathbf{F}}{A} = \hat{\mathbf{z}} \cdot \mathbf{T}\big|_{z=z_a-\delta}. \tag{5}$$

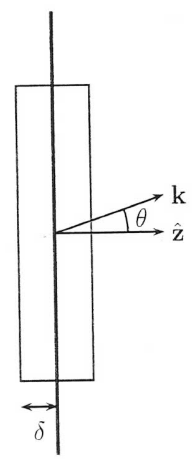

Figure 1: A plane wave with direction of propagation $\mathbf{k}$ incident on a screen.

where A is the total area of the screen. The electromagnetic stress tensor $\mathbf{T}$ in these expressions is given by

$$\mathbf{T} = \mathbf{1}U - (\mathbf{DE} + \mathbf{BH}), \tag{6}$$

where U is the electromagnetic energy density,

$$U = \frac{1}{2}(\mathbf{D} \cdot \mathbf{E} + \mathbf{B} \cdot \mathbf{H}). \tag{7}$$

(c) For the particular case when the plane wave is incident normally on the screen ($\theta = 0$ in Fig. 1) calculate the force per unit area in the direction normal to the screen by evaluating

$$\frac{\mathbf{F} \cdot \hat{\mathbf{z}}}{A}. \tag{8}$$

Express the answer in terms of U using the properties of a plane wave: $\mathbf{k} \cdot \mathbf{E} = 0, \mathbf{k} \cdot \mathbf{B} = 0, \mathbf{E} \cdot \mathbf{B} = 0, |\mathbf{E}| = c|\mathbf{B}|$, and $kc = \omega$.

(d) Consider the case when the plane wave is incident obliquely on the screen such that $\hat{\mathbf{k}} \cdot \hat{\mathbf{z}} = \cos\theta$ and $\mathbf{H} \cdot \hat{\mathbf{z}} = 0$. Calculate the force per unit area in the direction normal to the screen by evaluating

$$\frac{\mathbf{F} \cdot \hat{\mathbf{z}}}{A}, \tag{9}$$

and the force per unit area tangential to the screen by evaluating

$$\frac{\mathbf{F} \cdot \hat{\mathbf{x}}}{A}. \tag{10}$$

Express the answer in terms of U and θ using the properties of a plane wave.

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1. $$\vec{\nabla}(\hat{r}\cdot\vec{a}) = \vec{\nabla}\left[\frac{1}{r}(\vec{r}\cdot\vec{a})\right]$$

$$= \left(\vec{\nabla}\frac{1}{r}\right)(\vec{r}\cdot\vec{a}) + \frac{1}{r}(\vec{\nabla}\vec{r})\cdot\vec{a}$$

$$= -\frac{1}{r^2}(\vec{\nabla}r)(\vec{r}\cdot\vec{a}) + \frac{1}{r}\overleftrightarrow{I}\cdot\vec{a}$$

$$= -\frac{1}{r}\hat{r}(\hat{r}\cdot\vec{a}) + \frac{1}{r}\vec{a}$$

$$= -\frac{1}{r}\left[\hat{r}(\hat{r}\cdot\vec{a}) - \vec{a}\right]$$

$$= -\frac{1}{r}\hat{r}\times(\hat{r}\times\vec{a})$$

2. $$\vec{F} = q\left[\vec{E} + \vec{v}\times\vec{B}\right]$$

(a) $\vec{a} = 0 \Rightarrow \vec{F} = 0$

$$\vec{E} = -\vec{v}\times\vec{B} \qquad (\because \vec{v}\cdot\vec{B} = 0).$$

$$|\vec{E}| = v|\vec{B}|$$

Thus, $$v = \frac{|\vec{E}|}{|\vec{B}|}$$

(b) $$v^2 = \frac{E^2}{B^2} = \frac{E^2}{\mu_0^2 H^2} \qquad (\text{if } \mu_0 H^2 = \epsilon_0 E^2)$$

$$= \frac{E^2}{\mu_0\epsilon_0 E^2}$$

$$= c^2$$

$v = c$.

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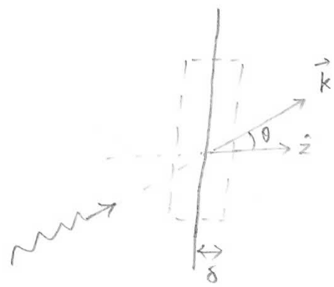


(a) $$\frac{\partial \vec{G}}{\partial t} + \vec{\nabla} \cdot \overleftrightarrow{T} + \vec{f} = 0$$

$$\int dx \int dy \int_{z_a-\delta}^{z_a+\delta} dz \, \frac{\partial \vec{G}}{\partial t} + \int dx\, dy \int_{z_a-\delta}^{z_a+\delta} dz \, \vec{\nabla} \cdot \overleftrightarrow{T} + \underbrace{\int dx\, dy \int_{z_a-\delta}^{z_a+\delta} dz \, \vec{f}}_{\vec{F}} = 0$$

Unless $\vec{G}$ has a δ-function contribution $\int_{z_a-\delta}^{z_a+\delta} dz \, \vec{G} = 0$.

$$\vec{F} = - \int_{-\infty}^{+\infty} dx \int_{-\infty}^{+\infty} dy \int_{z_a-\delta}^{z_a+\delta} dz \, \vec{\nabla} \cdot \overleftrightarrow{T}$$

(b) Using divergence theorem we have.

$$\vec{F} = - \oint d\vec{a} \cdot \overleftrightarrow{T}$$

$$= - A(-\hat{z}) \cdot \overleftrightarrow{T} \Big|_{z=z_a-\delta} - \underbrace{A(\hat{z}) \cdot \overleftrightarrow{T} \Big|_{z=z_a+\delta}}_{= 0 \ (\because \ E=0,\, B=0 \text{ on } z_a+\delta \text{ side})}$$

$$= A \ \hat{z} \cdot \overleftrightarrow{T} \Big|_{z=z_a-\delta}$$

Thus, $$\frac{\vec{F}}{A} = \hat{z} \cdot \overleftrightarrow{T} \Big|_{z=z_a-\delta}.$$

(c) $$\frac{\vec{F}\cdot\hat{z}}{A} = \hat{z}\cdot\overleftrightarrow{T}\cdot\hat{z}$$

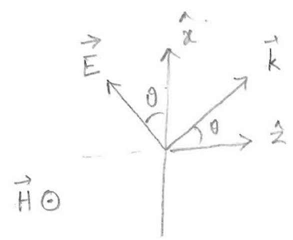
$$= U - (\hat{z}\cdot\vec{D})(\hat{z}\cdot\vec{E}) - (\hat{z}\cdot\vec{B})(\hat{z}\cdot\vec{H})$$

$$= U - \underbrace{(\hat{k}\cdot\vec{D})(\hat{k}\cdot\vec{E})}_{=0} - \underbrace{(\hat{k}\cdot\vec{B})(\hat{k}\cdot\vec{H})}_{=0}$$

$$= U$$

(d) $$\frac{\vec{F}\cdot\hat{z}}{A} = \hat{z}\cdot\overleftrightarrow{T}\cdot\hat{z}$$

$$= U - (\hat{z}\cdot\vec{D})(\hat{z}\cdot\vec{E}) - \underbrace{(\hat{z}\cdot\vec{B})}_{=0}(\hat{z}\cdot\vec{H})$$



$$E^2 = c^2 B^2 = \frac{\mu_0}{\epsilon_0} H^2.$$

$$\frac{\vec{F}\cdot\hat{z}}{A} = U - (-\epsilon_0 E \sin\theta)(-E\sin\theta) - 0$$

$$= U - \epsilon_0 E^2 \sin^2\theta.$$

$$= U \cos^2\theta$$

$$U = \tfrac{1}{2}\epsilon_0 E^2 + \tfrac{1}{2}\mu_0 H^2 = \epsilon_0 E^2.$$

$$\frac{\vec{F}\cdot\hat{x}}{A} = \hat{z}\cdot\overleftrightarrow{T}\cdot\hat{x}$$

$$= (\hat{z}\cdot\hat{x})U - (\hat{z}\cdot\vec{D})(\hat{x}\cdot\vec{E}) - \underbrace{(\hat{z}\cdot\vec{B})}_{=0}(\hat{z}\cdot\vec{H})$$

$$= 0 - (-\epsilon_0 E \sin\theta)(E\cos\theta) - 0$$

$$= U \sin\theta \cos\theta$$