SET - 20

Bessel functions

(1) The Green function, which is the electric potential of a unit point charge, satisfies

$$\epsilon_0 \nabla^2 G(\vec{r}, \vec{r}') = \delta^{(3)}(\vec{r} - \vec{r}') .$$

This is a Poisson equation, but satisfies the Laplace equation for $\vec{r} \neq \vec{r}'$.

(2) We have already learned:

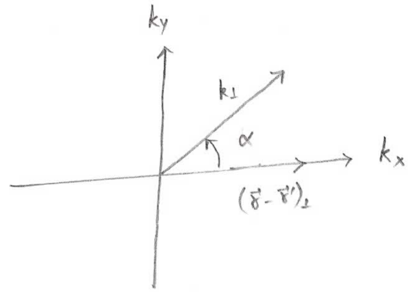
$$G(\vec{r}, \vec{r}') = \frac{1}{4\pi\epsilon_0} \frac{1}{|\vec{r} - \vec{r}'|} = \frac{1}{\epsilon_0} \int \frac{d^3k}{(2\pi)^3} e^{i\vec{k}\cdot(\vec{r}-\vec{r}')} \frac{1}{k^2}$$

$$= \frac{1}{\epsilon_0} \int \frac{d^2k_\perp}{(2\pi)^2} e^{i\vec{k}_\perp \cdot (\vec{r}-\vec{r}')_\perp} \frac{1}{2k_\perp} e^{-k_\perp |z-z'|} .$$

The second equality was derived in the context of Green's function in free space and the third equality was derived in the context of Green's function in the presence of configurations with planar geometries.

(3) Consider the $\vec{k}_\perp$ - plane

$$\vec{k}_\perp = (k_x, k_y) = (k_\perp, \alpha)$$



choose $(\vec{r}-\vec{r}')_\perp$ along k_x.

$$\vec{k}_\perp \cdot (\vec{r}-\vec{r}')_\perp = k_\perp P \cos\alpha$$

where $|(\vec{r}-\vec{r}')_\perp| = P$.

(4) Using (3) in (2)

$$G(\vec{r},\vec{r}') = \frac{1}{\epsilon_0}\int \frac{d^2k_\perp}{(2\pi)^2}\, e^{i\vec{k}_\perp\cdot(\vec{r}-\vec{r}')_\perp}\, \frac{1}{2k_\perp}\, e^{-k_\perp|z-z'|}$$

$$= \frac{1}{\epsilon_0}\int_0^\infty \frac{\cancel{k_\perp}\, dk_\perp}{2\pi}\int_0^{2\pi}\frac{d\alpha}{2\pi}\, e^{ik_\perp P\cos\alpha}\,\frac{1}{2\cancel{k_\perp}}\, e^{-k_\perp|z-z'|}$$

$$= \frac{1}{4\pi\epsilon_0}\int_0^\infty dk_\perp \boxed{\int_0^{2\pi}\frac{d\alpha}{2\pi}\, e^{ik_\perp P\cos\alpha}}\; e^{-ik_\perp|z-z'|}$$

(5) Bessel function of zeroth order is defined using the integral representation

$$J_0(t) = \int_0^{2\pi}\frac{d\alpha}{2\pi}\, e^{it\cos\alpha}$$

6 We verify that $J_0(t)$ is a real function:

$$J_0(t) = \int_0^{2\pi} \frac{d\alpha}{2\pi} \left[\cos(t \cos\alpha) + i \sin(t\cos\alpha) \right]$$

$$\int_0^{2\pi} \frac{d\alpha}{2\pi} \sin(t\cos\alpha) = \int_0^{\pi} \frac{d\alpha}{2\pi} \sin(t\cos\alpha) + \underbrace{\int_{\pi}^{2\pi} \frac{d\alpha}{2\pi} \sin(t\cos\alpha)}_{\downarrow\ \alpha' = \alpha - \pi}$$

$$= \int_0^{\pi} \frac{d\alpha}{2\pi} \sin(t\cos\alpha) + \int_0^{\pi} \frac{d\alpha'}{2\pi} \sin\left(t\cos(\pi+\alpha')\right)$$

$$= \int_0^{\pi} \frac{d\alpha}{2\pi} \sin(t\cos\alpha) - \int_0^{\pi} \frac{d\alpha'}{2\pi} \sin(t\cos\alpha')$$

$$= 0$$

$$\Rightarrow \quad \mathrm{Im}\left[J_0(t)\right] = 0$$

7 Thus,

$$J_0(t) = \int_0^{2\pi} \frac{d\alpha}{2\pi} \cos(t\cos\alpha)$$

which immediately implies

$$J_0(-t) = J_0(t).$$

8 Using the second equality of 2 in 1 we learn that, letting $\vec{r} \neq \vec{r}\,' = 0$ for simplicity,

$$\int d^3k \, \frac{1}{k^2} \, \nabla^2 \left[e^{i k_x x} \, e^{i k_y y} \, e^{i k_z z} \right] \cancel{\times} \cancel{\delta} = \cancel{\delta^{(3)}(\vec{x})}.$$
$$= 0$$

~~if we have.~~

$$\cancel{k_x^2 + k_y^2 + k_z^2 = 0.}$$

9 Similarly, using the third equality of 2 in 1 we learn that, $\vec{r} \neq \vec{r}\,' = 0$, $|\vec{r}_\perp| = \rho$

$$\nabla^2 \left[J_0 (k_\perp \rho) \, e^{-k_\perp |z|} \right] = 0$$

10 Using

$$\vec{\nabla} = \hat{\rho} \frac{\partial}{\partial \rho} + \hat{\phi} \frac{1}{\rho} \frac{\partial}{\partial \phi} + \hat{z} \frac{\partial}{\partial z}$$

$$\nabla^2 = \frac{1}{\rho} \frac{\partial}{\partial \rho} \rho \frac{\partial}{\partial \rho} + \frac{1}{\rho^2} \frac{\partial^2}{\partial \phi^2} + \frac{\partial^2}{\partial z^2}$$

and ignoring a $\delta(z)$ contribution, because $\vec{r} \neq \vec{r}\,' = 0$,

$$\left[\frac{1}{\rho} \frac{\partial}{\partial \rho} \rho \frac{\partial}{\partial \rho} + k_\perp^2 \right] J_0 (k_\perp \rho) = 0 \qquad t = k_\perp \rho$$

$$\left[\frac{1}{t} \frac{d}{dt} t \frac{d}{dt} + 1 \right] J_0 (t) = 0$$

$$\left[\frac{d^2}{dt^2} + \frac{1}{t} \frac{d}{dt} + 1 \right] J_0 (t) = 0 .$$

11 Using 5 and 10 together, we verify

$$\left[\frac{d^2}{dt^2} + \frac{1}{t}\frac{d}{dt} + 1\right] \int_0^{2\pi} \frac{d\alpha}{2\pi}\, e^{it\cos\alpha}$$

$$= \int_0^{2\pi} \frac{d\alpha}{2\pi} \left[(i\cos\alpha)^2 + \frac{1}{t}\, i\cos\alpha + 1 \right] e^{it\cos\alpha}$$

$$= \int_0^{2\pi} \frac{d\alpha}{2\pi} \left[\sin^2\alpha + \frac{1}{t}\, i\cos\alpha \right] e^{it\cos\alpha}$$

$$= \int_0^{2\pi} \frac{d\alpha}{2\pi} \frac{d}{d\alpha} \left[\frac{i}{t} \sin\alpha\, e^{it\cos\alpha} \right]$$

$$= \frac{1}{2\pi} \frac{i}{t} \sin\alpha\, e^{it\cos\alpha} \Big|_{\alpha=0}^{\alpha=2\pi}$$

$$= 0 \quad ✓$$

12 Notice that the solutions in 9 do not have any dependence in the polar angle ϕ. We shall next explore this dependence.

(13) Bessel function of m-th order is defined using the generating function

$$e^{it\cos\alpha} = e^{it\frac{(e^{i\alpha}+e^{-i\alpha})}{2}} = e^{\frac{t}{2}\left(u-\frac{1}{u}\right)} \qquad u = i e^{i\alpha}$$

using

$$e^{\frac{t}{2}\left(u-\frac{1}{u}\right)} = \sum_{m=-\infty}^{+\infty} u^m J_m(t)$$

(14)

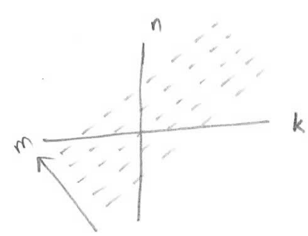
$$e^{\frac{t}{2}\left(u-\frac{1}{u}\right)} = e^{\frac{t}{2}u}\, e^{-\frac{t}{2u}}$$

$$= \sum_{n=0}^{\infty} \frac{1}{n!}\left(u\frac{t}{2}\right)^n \sum_{k=0}^{\infty} \frac{(-1)^k}{k!}\left(\frac{t}{2u}\right)^k$$

$$= \sum_{n=0}^{\infty}\sum_{k=0}^{\infty} \frac{(-1)^k}{n!\,k!}\left(\frac{t}{2}\right)^{n+k} u^{n-k} \qquad n = k+m.$$

$m = n-k$

$k = k$



$-\infty < m < \infty$

$$= \sum_{m=-\infty}^{+\infty}\sum_{k=0}^{\infty} \frac{(-1)^k}{k!\,(k+m)!}\left(\frac{t}{2}\right)^{m+2k} u^m$$

$$= \sum_{m=-\infty}^{+\infty} u^m \underbrace{\sum_{k=0}^{\infty} \frac{(-1)^k}{k!\,(k+m)!}\left(\frac{t}{2}\right)^{m+2k}}_{J_m(t)}$$

15 Using 13 and 14 together we have

$$e^{it\cos\alpha} = \sum_{m=-\infty}^{+\infty} i^m e^{im\alpha} J_m(t)$$

$$\int_0^{2\pi} \frac{d\alpha}{2\pi} e^{-im'\alpha} e^{it\cos\alpha} = \sum_{m=-\infty}^{+\infty} i^m J_m(t) \underbrace{\int_0^{2\pi} \frac{d\alpha}{2\pi} e^{im\alpha} e^{-im'\alpha}}_{\delta_{mm'}}$$

$$\Rightarrow \quad i^m J_m(t) = \int_0^{2\pi} \frac{d\alpha}{2\pi} e^{it\cos\alpha - im\alpha}$$

16 Observing the symmetry under the substitution

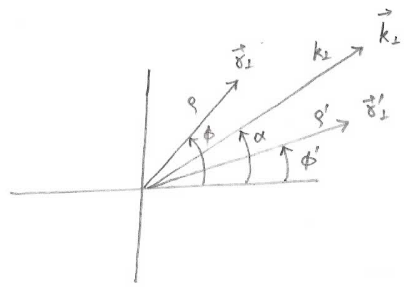
$$u \to -\frac{1}{u}$$

on the left hand side of 13 we learn

$$J_{-m}(t) = (-1)^m J_m(t)$$

(17) Let us next consider the δ-function in two dimensions:

$$\delta(x-x')\,\delta(y-y') = \int \frac{d^2k_\perp}{(2\pi)^2}\, e^{i\vec{k}_\perp \cdot (\vec{r}-\vec{r}')_\perp}$$



$$\vec{k}_\perp = (k_\perp, \alpha)$$

$$\vec{r}_\perp = (\rho, \phi) = (x, y)$$

$$\vec{r}'_\perp = (\rho', \phi') = (x', y')$$

$$\vec{k}_\perp \cdot \vec{r}_\perp = k_\perp \rho \cos(\phi - \alpha)$$

$$\vec{k}_\perp \cdot \vec{r}'_\perp = k_\perp \rho' \cos(\phi' - \alpha)$$

$$\frac{\delta(\rho-\rho')}{\rho}\,\delta(\phi-\phi') = \int_0^\infty k_\perp\, dk_\perp \frac{1}{2\pi} \int_0^{2\pi} \frac{d\alpha}{2\pi}\, e^{i k_\perp \rho \cos(\phi-\alpha)}\, e^{-i k_\perp \rho' \cos(\phi'-\alpha)}$$

$$= \int_0^\infty k_\perp\, dk_\perp \frac{1}{2\pi} \int_0^{2\pi} \frac{d\alpha}{2\pi} \sum_{m=-\infty}^{+\infty} i^m e^{im(\phi-\alpha)} J_m(k_\perp \rho) \sum_{m'=-\infty}^{+\infty} (-i)^{m'} e^{-im'(\phi'-\alpha)} J_{m'}(k_\perp \rho')$$

$$= \int_0^\infty k_\perp\, dk_\perp \frac{1}{2\pi} \sum_{m=-\infty}^{+\infty} \sum_{m'=-\infty}^{+\infty} i^m (-i)^{m'} e^{im\phi - im'\phi'} J_m(k_\perp \rho)\, J_{m'}(k_\perp \rho')$$

$$\times \int_0^{2\pi} \frac{d\alpha}{2\pi}\, e^{i(m'-m)\alpha}$$

$$= \int_0^\infty k_\perp\, dk_\perp \frac{1}{2\pi} \sum_{m=-\infty}^{+\infty} \sum_{m'=-\infty}^{+\infty} i^m (-i)^{m'} e^{im\phi - im'\phi'} J_m(k_\perp \rho)\, J_{m'}(k_\perp \rho')\, \delta_{mm'}$$

$$= \int_0^\infty k_\perp\, dk_\perp \frac{1}{2\pi} \sum_{m=-\infty}^{+\infty} e^{im(\phi-\phi')} J_m(k_\perp \rho)\, J_m(k_\perp \rho')$$

$$= \frac{1}{2\pi} \sum_{m=-\infty}^{+\infty} e^{im(\phi-\phi')} \int_0^\infty k_\perp\, dk_\perp\, J_m(k_\perp \rho)\, J_m(k_\perp \rho')$$

$$\Rightarrow \quad \frac{\delta(\rho-\rho')}{\rho} = \int_0^\infty k_\perp\, dk_\perp\, J_m(k_\perp \rho)\, J_m(k_\perp \rho') \quad \longrightarrow \text{completeness relation of Bessel functions.}$$

18 Orthogonality :

$$\frac{\delta(k-k')}{k} = \int_0^\infty \rho\, d\rho \; J_m(k\rho)\; J_m(k'\rho)$$

The appearance of m on right hand side is a reminder that $J_m(k_\perp \rho)$ and $e^{im\phi}$ are coupled together by m in ρ and ϕ.

19 Using (17) in (4) and (2) we have.

$$G(\vec{r},\vec{r}') = \frac{1}{4\pi\epsilon_0}\frac{1}{|\vec{r}-\vec{r}'|} = \frac{1}{\epsilon_0}\int \frac{d^3k}{(2\pi)^3}\, e^{i\vec{k}\cdot(\vec{r}-\vec{r}')}\,\frac{1}{k^2}$$

$$= \frac{1}{\epsilon_0}\int \frac{d^2k_\perp}{(2\pi)^2}\, e^{i\vec{k}_\perp\cdot(\vec{r}-\vec{r}')_\perp}\,\frac{1}{2k_\perp}\, e^{-k_\perp|z-z'|}$$

$$= \frac{1}{4\pi\epsilon_0}\int_0^\infty dk_\perp\, J_0(k_\perp P)\, e^{-k_\perp|z-z'|} \qquad P = |(\vec{r}-\vec{r}')_\perp|$$

$$= \frac{1}{4\pi\epsilon_0}\int_0^\infty dk_\perp \sum_{m=-\infty}^{+\infty} J_m(k_\perp\rho)\, J_m(k_\perp\rho')\, e^{im(\phi-\phi')}\, e^{-k_\perp|z-z'|}$$

20 Addition theorem: Fourth and fifth equality in 19 leads to

$$J_0(k_\perp P) = \sum_{m=-\infty}^{+\infty} e^{im(\phi-\phi')} J_m(k_\perp \rho) J_m(k_\perp \rho')$$

where

$$P = |(\vec{r}-\vec{r}')_\perp| = \sqrt{\rho^2 + \rho'^2 - 2\rho\rho' \cos(\phi-\phi')} .$$

21 Using the fifth equality in 19 as a solution to the Laplacian in 1, $\vec{r} \neq \vec{r}' = 0$, we learn

$$\nabla^2 \left[J_m(k_\perp \rho)\, e^{im\phi}\, e^{-k_\perp |z|} \right] = 0$$

$$\left[\frac{1}{\rho}\frac{\partial}{\partial\rho}\rho\frac{\partial}{\partial\rho} - \frac{m^2}{\rho^2} + k_\perp^2 \right] J_m(k_\perp \rho) = 0$$

$$\left[\frac{1}{t}\frac{d}{dt} t \frac{d}{dt} - \frac{m^2}{t^2} + 1 \right] J_m(t) = 0 \qquad t = k_\perp \rho$$

$$\left[\frac{d^2}{dt^2} + \frac{1}{t}\frac{d}{dt} - \frac{m^2}{t^2} + 1 \right] J_m(t) = 0 .$$

22 To identify the asymptotic forms we we

$$\frac{1}{\sqrt{t}} \frac{d}{dt} \sqrt{t} = \left(\frac{1}{2t} + \frac{d}{dt}\right)$$

$$\left(\frac{1}{\sqrt{t}} \frac{d}{dt} \sqrt{t}\right)^2 = \frac{1}{\sqrt{t}} \frac{d^2}{dt^2} \sqrt{t} = \left(\frac{1}{2t} + \frac{d}{dt}\right)^2$$

$$= \frac{d^2}{dt^2} + \frac{1}{t}\frac{d}{dt} - \frac{1}{4t^2}$$

23 Using 22 in 21

$$\left[\frac{1}{\sqrt{t}} \frac{d^2}{dt^2} \sqrt{t} - \frac{(m^2 - \frac{1}{4})}{t^2} + 1\right] J_m(t) = 0$$

$$\left[\frac{d^2}{dt^2} - \frac{(m^2 - \frac{1}{4})}{t^2} + 1\right] \sqrt{t}\, J_m(t) = 0$$

24 For $t \to 0$, such that $|m^2 - \frac{1}{4}| << |t|$,

$$\left[\frac{d^2}{dt^2} - \frac{(m^2 - \frac{1}{4})}{t^2}\right] \sqrt{t}\, J_m(t) = 0$$

$$\Rightarrow \quad \sqrt{t}\, J_m(t) \sim t^{|m| + \frac{1}{2}}$$

$$J_m(t) \sim t^{|m|}$$

Using 14 : $J_m(t) = \frac{1}{m!}\left(\frac{t}{2}\right)^m + \cdots \qquad m \geq 0, \ t \to 0.$

(25) For $t \to \infty$, such that $|m^2 - \frac{1}{4}| >> |t|$

$$\left[\frac{d^2}{dt^2} + 1\right] \sqrt{t}\ J_m(t) = 0$$

$$\Rightarrow \quad \sqrt{t}\ J_m(t) \sim \cos t \text{ or } \sin t$$

Correct form is :

$$J_m(t) \sim \sqrt{\frac{2}{\pi t}}\ \cos\left(t - \left(m + \tfrac{1}{2}\right)\frac{\pi}{2}\right).$$