SET - 19

Fourier space

1. Vectors

$$\hat{e}_1^{(j)} = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} \downarrow j \qquad \hat{e}_2^{(j)} = \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}$$

Orthogonality:

$$\sum_{j=1}^{N} \hat{e}_m^{(j)} \cdot \hat{e}_{m'}^{(j)} = \delta_{mm'}$$

Completeness:

Any vector can be written in terms of the eigenvector

$$\vec{A}^{(j)} = a_1 \hat{e}_1^{(j)} + a_2 \hat{e}_2^{(j)} + \dots$$

$$= \sum_{m=1}^{N} \hat{e}_m^{(j)} a_m$$

The completeness relation can also be stated as

$$\overleftrightarrow{1}^{ij} = \vec{a}_1^{\,i} \hat{e}_1^{j} + \vec{a}_2^{\,i} \hat{e}_2^{j} + \dots$$

$$= \sum_{m=1}^{N} \vec{a}_m^{\,i} \hat{e}_m^{j}$$

$$\overleftrightarrow{1}^{ij} \cdot \hat{e}_{m'}^{j} = \sum_{m=1}^{N} \vec{a}_m^{\,i} \delta_{mm'}$$

$$\vec{a}_m^{\,i} = \hat{e}_{m'}^{i}$$

$$\Rightarrow \quad \overleftrightarrow{1}^{ij} = \sum_{m=1}^{N} \hat{e}_m^{i} \hat{e}_m^{j}$$

2 Fourier space : eigenvectors are $\boxed{e^{im\phi}}$

$0 \leq \phi < 2\pi$ $\qquad m = 0, \pm 1, \pm 2, \ldots$

Orthogonality :

$$\int_0^{2\pi} \frac{d\phi}{2\pi} \; e^{im\phi} \, e^{-im'\phi} = \delta_{mm'}$$

because for $m = m'$ we have

$$\int_0^{2\pi} \frac{d\phi}{2\pi} = 1$$

and for $m \neq m'$ the integral is zero because

$$\int_0^{2\pi} \frac{d\phi}{2\pi} \cos(m-m')\phi = 0 \quad \text{and} \quad \int_0^{2\pi} \frac{d\phi}{2\pi} \sin(m-m')\phi = 0 .$$

Completeness :

Any function, with periodic boundary conditions, $f(2\pi) = f(0)$, can be written in terms of the Fourier functions,

$$f(\phi) = \frac{1}{2\pi} \sum_{m=-\infty}^{+\infty} e^{im\phi} \tilde{f}_m ,$$

which can also be expressed in terms of δ-function as.

$$\delta(\phi - \phi') = \frac{1}{2\pi} \sum_{m=-\infty}^{+\infty} e^{im(\phi-\phi')} d_m .$$

To find d_m we use the orthogonality relation

$$\int_0^{2\pi} d(\phi-\phi')\, e^{-im'(\phi-\phi')}\, \delta(\phi-\phi')$$

$$= \int_0^{2\pi} d(\phi-\phi')\, e^{-im'(\phi-\phi')}\, \frac{1}{2\pi} \sum_{m=-\infty}^{+\infty} e^{im(\phi-\phi')}\, d_m$$

$$1 = \sum_{m=-\infty}^{+\infty} d_m\, \delta_{mm'}$$

$$d_m = 1.$$

Thus, the statement of completeness is encoded in

$$\delta(\phi-\phi') = \frac{1}{2\pi} \sum_{m=-\infty}^{+\infty} e^{-im\phi}\, e^{im\phi}$$

3 For functions defined on the complete real line.

we use

$$e^{im\phi} \rightarrow e^{im 2\pi \frac{x}{L}} \rightarrow e^{ikx} \qquad \frac{2\pi m}{L} \rightarrow k.$$

Orthogonality:

$$\int_{-\infty}^{+\infty} dx\, e^{ikx}\, e^{-ik'x} = 2\pi\, \delta(k-k')$$

Completeness:

$$f(x) = \int_{-\infty}^{+\infty} \frac{dk}{2\pi}\, e^{ikx}\, \tilde{f}(k)$$

or, using orthogonality

$$\delta(x-x') = \int_{-\infty}^{+\infty} \frac{dk}{2\pi}\, e^{-ikx}\, e^{ikx'}$$