Method of images

(1) Let us consider the electric potential due to a point charge, q, in vacuum.

$$-\vec{\nabla}\cdot\epsilon_0\vec{\nabla}\phi(\vec{r}) = \rho(\vec{r}) = q\,\delta^{(3)}(\vec{r}-\vec{r}')$$

$$-\vec{\nabla}\cdot\epsilon_0\vec{\nabla}G_0(\vec{r},\vec{r}') = \delta^{(3)}(\vec{r}-\vec{r}')$$

$$\begin{aligned}\phi(\vec{r}) &= \int d^3r'\,G_0(\vec{r},\vec{r}')\,\rho(\vec{r}') \\ &= q\int d^3r'\,G_0(\vec{r},\vec{r}')\,\delta^{(3)}(\vec{r}-\vec{r}') \\ &= q\,G_0(\vec{r},\vec{r}')\end{aligned}$$

Thus, electric potential due to a unit point charge is the Green's function.

(2) Since we know that the electric potential due to unit point charge in vacuum is given by

$$\frac{1}{4\pi\epsilon_0}\frac{1}{|\vec{r}-\vec{r}'|}$$

we identify the integral representation for the Green's function to be

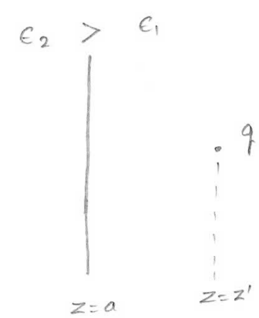
$$G_0(\vec{r},\vec{r}') = \frac{1}{4\pi\epsilon_0}\frac{1}{|\vec{r}-\vec{r}'|} = \int\frac{d^2k_\perp}{(2\pi)^2}\,e^{i\vec{k}_\perp\cdot(\vec{r}-\vec{r}')_\perp}\,\frac{1}{\epsilon_0}\frac{1}{2k_\perp}\,e^{-k_\perp|z-z'|}.$$

(3) More explicitly we have $\qquad k_\perp = \sqrt{k_x^2 + k_y^2}$

$$G_0(\vec{r}, \vec{r}\,') = \frac{1}{4\pi\epsilon_0} \frac{1}{\sqrt{(x-x')^2 + (y-y')^2 + (z-z')^2}}$$

$$= \int_{-\infty}^{+\infty} \frac{dk_x}{2\pi} \int_{-\infty}^{+\infty} \frac{dk_y}{2\pi} \, e^{ik_x(x-x')} e^{ik_y(y-y')} \frac{1}{\epsilon_0} \frac{1}{2k_\perp} e^{-k_\perp|z-z'|}$$

(4) Let us now determine the electric potential due to a point charge, q, placed on the right hand side of the interface of two dielectric. Assume $\epsilon_2 > \epsilon_1$. ϵ



(5) Using the solution for Green's function (for $a<z'$) we have.

$$\phi(\vec{r}) = q\, G(\vec{r}, \vec{r}\,')$$

$$= q \int \frac{d^2k_\perp}{(2\pi)^2} e^{i\vec{k}_\perp\cdot(\vec{r}-\vec{r}\,')_\perp} g(z, z'; k_\perp)$$

$$= q \int \frac{d^2k_\perp}{(2\pi)^2} e^{i\vec{k}_\perp\cdot(\vec{r}-\vec{r}\,')_\perp} \left[\frac{1}{\epsilon_1}\frac{1}{2k_\perp} e^{-k_\perp|z-z'|} - \frac{1}{\epsilon_1}\frac{1}{2k_\perp}\frac{\epsilon_2-\epsilon_1}{\epsilon_2+\epsilon_1} e^{-k_\perp|z-a|} e^{-k_\perp|z'-a|} \right]$$

(6) Using (3) we have

$$\phi(\vec{r}) = \frac{1}{4\pi\epsilon_1} q \frac{1}{\sqrt{(x-x')^2 + (y-y')^2 + (z-z')^2}}$$

$$- \frac{1}{4\pi\epsilon_1} q \frac{\epsilon_2-\epsilon_1}{\epsilon_2+\epsilon_1} \frac{1}{\sqrt{(x-x')^2 + (y-y')^2 + (|z-a| + |z'-a|)^2}}$$

$$= \frac{1}{4\pi\epsilon_1} q \frac{1}{|\vec{r}-\vec{r}'|} + \frac{1}{4\pi\epsilon_1} q_{image} \frac{1}{|\vec{r} - \vec{r}'_{image}|}$$

Note: $\epsilon_2 > \epsilon_1$, $a < z'$.

where

$$q_{image} = - q \frac{\epsilon_2-\epsilon_1}{\epsilon_2+\epsilon_1}$$

$$\vec{r}'_{image} = \begin{cases} (x', y', 2a - z') & \text{if } a < z, \\ (x', y', z') & \text{if } z < a. \end{cases}$$

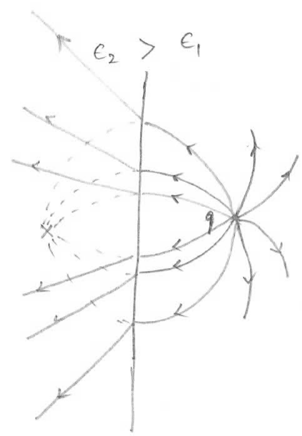
(7) Notice that we have

$$\phi(\vec{r}) = \begin{cases} \frac{1}{4\pi\epsilon_1} q \frac{1}{\sqrt{(x-x')^2 + (y-y')^2 + (z-z')^2}} - \frac{1}{4\pi\epsilon_1} q \frac{\epsilon_2-\epsilon_1}{\epsilon_2+\epsilon_1} \frac{1}{\sqrt{(x-x')^2 + (y-y')^2 + (z+z'-2a)^2}}, & a < z \\ \frac{1}{4\pi(\epsilon_2+\epsilon_1)} 2q \frac{1}{\sqrt{(x-x')^2 + (y-y')^2 + (z-z')^2}} & z < a \end{cases}$$

(8) The electric field is

$$\vec{E}(\vec{r}) = - \vec{\nabla} \phi(\vec{r})$$

$$= \frac{q}{4\pi \epsilon_1} \frac{\vec{r} - \vec{r}'}{|\vec{r} - \vec{r}'|^3} + \frac{q_{image}}{4\pi \epsilon_1} \frac{\vec{r} - \vec{r}'_{image}}{|\vec{r} - \vec{r}''|^3}$$



(9) In the homework you will determine the electric field for $\epsilon_2 < \epsilon_1$.

10 A conductor is a dielectric material with a very high ($\epsilon \to \infty$) dielectric constant. Thus,

$$q_{image} = \lim_{\epsilon_2 \to \infty} -q \frac{\epsilon_2 - \epsilon_1}{\epsilon_2 + \epsilon_1}$$

$$= -q$$

Thus,

$$\phi(\vec{r}) = \begin{cases} \dfrac{q}{4\pi\epsilon_1} \dfrac{1}{\sqrt{(x-x')^2 + (y-y')^2 + (z-z')^2}} - \dfrac{q}{4\pi\epsilon_1} \dfrac{1}{\sqrt{(x-x')^2 + (y-y')^2 + (z+z'-2a)^2}}, & a < z \\ 0, & z < a. \end{cases}$$

$$\vec{E}(\vec{r}) = \begin{cases} \dfrac{q}{4\pi\epsilon_1} \dfrac{\vec{r} - \vec{r}'}{|\vec{r} - \vec{r}'|} - \dfrac{q}{4\pi\epsilon_1} \dfrac{\vec{r} - \vec{r}'_{image}}{|\vec{r} - \vec{r}'_{image}|} & a < z \\ 0, & z < a. \end{cases}$$

(11) Let us investigate the continuity conditions on the components of the electric field.

$$\vec{E}(\vec{r}) = \begin{cases} \dfrac{q}{4\pi\epsilon_1} \dfrac{(x-x')\,\hat{i} + (y-y')\,\hat{j} + (z-z')\,\hat{k}}{\left[\sqrt{(x-x')^2 + (y-y')^2 + (z-z')^2}\right]^3} \\[2ex] \quad - \dfrac{q}{4\pi\epsilon_1} \dfrac{\epsilon_2-\epsilon_1}{\epsilon_2+\epsilon_1} \dfrac{(x-x')\,\hat{i} + (y-y')\,\hat{j} + (z+z'-2a)\,\hat{k}}{\left[\sqrt{(x-x')^2 + (y-y')^2 + (z+z'-2a)^2}\right]^3}, & a <\,, z, z' \\[2ex] \dfrac{2q}{4\pi(\epsilon_2+\epsilon_1)} \dfrac{(x-x')\,\hat{i} + (y-y')\,\hat{j} + (z-z')\,\hat{k}}{\left[\sqrt{(x-x')^2 + (y-y')^2 + (z-z')^2}\right]^3}, & z < a < z' \end{cases}$$

(12)

$$E_x(x, y, a+\delta) = \frac{2q}{4\pi(\epsilon_2+\epsilon_1)} \frac{(x-x')}{\left[\sqrt{(x-x')^2 + (y-y')^2 + (a-z')^2}\right]^3}$$

$$E_x(x, y, a-\delta) = \frac{2q}{4\pi(\epsilon_2+\epsilon_1)} \frac{x-x'}{\left[\sqrt{(x-x')^2 + (y-y')^2 + (a-z')^2}\right]^3}$$

$$E_y(x, y, a+\delta) = \frac{2q}{4\pi(\epsilon_2+\epsilon_1)} \frac{(y-y')}{\left[\sqrt{(x-x')^2 + (y-y') + (a-z')^2}\right]^3}$$

$$E_y(x, y, a-\delta) = \frac{2q}{4\pi(\epsilon_2+\epsilon_1)} \frac{(y-y')}{\left[\sqrt{(x-x')^2 + (y-y')^2 + (a-z')^2}\right]^3}$$

$$E_z(x, y, a+\delta) = \frac{q}{4\pi\epsilon_1} \frac{(a-z')}{\left[\sqrt{(x-x')^2 + (y-y')^2 + (a-z')^2}\right]^3} - \frac{q}{4\pi\epsilon_1} \frac{\epsilon_2-\epsilon_1}{\epsilon_2+\epsilon_1} \frac{(z'-a)}{\left[\sqrt{(x-x')^2 + (y-y')^2 + (a-z')^2}\right]^3}$$

$$= \frac{\epsilon_2}{\epsilon_1} \frac{2q}{4\pi(\epsilon_2+\epsilon_1)} \frac{(a-z')}{\left[\sqrt{(x-x')^2 + (y-y')^2 + (a-z')^2}\right]^3}$$

$$E_z(x, y, a-\delta) = \frac{2q}{4\pi(\epsilon_2+\epsilon_1)} \frac{(a-z')}{\left[\sqrt{(x-x')^2 + (y-y')^2 + (a-z')^2}\right]^3}$$

13 Using 12 we conclude the continuity conditions on electric field at a.

$$E_x(x, y, a+\delta) = E_x(x, y, a-\delta)$$

$$E_y(x, y, a+\delta) = E_y(x, y, a-\delta)$$

$$\epsilon_1 E_z(x, y, a+\delta) = \epsilon_2 E_z(x, y, a-\delta)$$

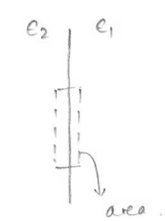
14 These continuity conditions, of course, could have been arrived at directly from the Maxwell's equations,

$$\vec{\nabla} \cdot \left[\epsilon(\vec{r}) \vec{E}(\vec{r})\right] = \rho(\vec{r})$$

Note that the induced charges in the form of polarization vector is absorbed into $\epsilon(\vec{r})$

$$\epsilon(\vec{r}) \vec{E}(\vec{r}) = \epsilon_0 \vec{E}(\vec{r}) + \vec{P}(\vec{r}).$$

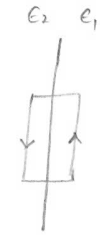
Using the divergence theorem on the Maxwell's equat



$$\oint d\vec{a} \cdot \left[\epsilon(\vec{r}) \vec{E}(\vec{r})\right] = 0$$

$$\Rightarrow \quad \epsilon_1 E_z(x, y, a+\delta) = \epsilon_2 E_z(x, y, a-\delta).$$

15 To derive the continuity in the other components we use



$$\vec{\nabla} \times \vec{E} = 0$$

$$\oint d\vec{l} \cdot \vec{E} = 0$$

$$\Rightarrow \quad E_x(x, y, a+\delta) = E_x(x, y, a-\delta)$$

$$E_y(x, y, a+\delta) = E_y(x, y, a-\delta)$$

16 For a perfect conductor, using 12, we have

$$\left.\begin{aligned} E_x(x, y, a+\delta) &= E_x(x, y, a-\delta) = 0 \\ E_y(x, y, a+\delta) &= E_y(x, y, a-\delta) = 0 \\ E_z(x, y, a+\delta) &= \frac{2q}{4\pi\epsilon_1} \frac{(a-z')}{\left[\sqrt{(x-x')^2 + (y-y')^2 + (z-z')^2}\right]^3} \\ E_z(x, y, a-\delta) &= 0 \end{aligned}\right\} \text{ for } \epsilon_2 \to \infty .$$