Green's function - Introduction

(1) Electrostatics is governed by

$$\vec{\nabla}\cdot(\epsilon_0 \vec{E}) = \rho - \vec{\nabla}\cdot\vec{P}$$

and

$$\vec{\nabla}\times\vec{E} = 0 \quad \Rightarrow \quad \vec{E} = -\vec{\nabla}\phi$$

(2) With the definition

$$\epsilon(\vec{r})\,\vec{E}(\vec{r}) = \epsilon_0\,\vec{E}(\vec{r}) + \vec{P}(\vec{r})$$

(3) the equations for electrostatics is

$$\vec{\nabla}\cdot\left[\epsilon(\vec{r})\,\vec{E}(\vec{r})\right] = \rho(\vec{r}),$$

(4) which can be expressed in terms of the electric potential $\phi(\vec{r})$ as

$$-\vec{\nabla}\cdot\left[\epsilon(\vec{r})\,\vec{\nabla}\phi(\vec{r})\right] = \rho(\vec{r}).$$

(5) The Green's function is defined as the electric potential due to a unit point charge,

$$-\vec{\nabla}\cdot\left[\epsilon(\vec{r})\,\vec{\nabla}\,G(\vec{r},\vec{r}\,')\right] = \delta^{(3)}(\vec{r}-\vec{r}\,'),$$

where $\vec{r}\,'$ is the position of the point charge.

6 The Green's function remains unchanged under the swap between the source point $\vec{r}'$ and observation point $\vec{r}$. That is,

$$G(\vec{r}, \vec{r}') = G(\vec{r}', \vec{r}).$$

This is called the reciprocity relation.

7 To prove the reciprocity relation we write the Green's function equation for $\vec{r}'$ and $\vec{r}''$:

$$-\vec{\nabla}\cdot\left[\epsilon(\vec{r})\,\vec{\nabla}G(\vec{r}, \vec{r}')\right] = \delta^{(3)}(\vec{r}-\vec{r}') \quad \text{— (i)}$$

$$-\vec{\nabla}\cdot\left[\epsilon(\vec{r})\,\vec{\nabla}G(\vec{r}, \vec{r}'')\right] = \delta^{(3)}(\vec{r}-\vec{r}'') \quad \text{— (ii)}$$

8 Multiplying 7-(i) by $G(\vec{r}, \vec{r}'')$, 7-(ii) by $G(\vec{r}, \vec{r}')$, and integrating both equations with respect to $\vec{r}$, we have

$$-\int d^3r\, G(\vec{r}, \vec{r}'')\,\vec{\nabla}\cdot\left[\epsilon(\vec{r})\,\vec{\nabla}G(\vec{r}, \vec{r}')\right] = \int d^3r\, G(\vec{r}, \vec{r}'')\,\delta^{(3)}(\vec{r}-\vec{r}') \quad \text{—(i)}$$

$$-\int d^3r\, G(\vec{r}, \vec{r}')\,\vec{\nabla}\cdot\left[\epsilon(\vec{r})\,\vec{\nabla}G(\vec{r}, \vec{r}'')\right] = \int d^3r\, G(\vec{r}, \vec{r}')\,\delta^{(3)}(\vec{r}-\vec{r}'') \quad \text{—(ii)}$$

9 Evaluating the integrals on the right hand side in 8 and subtracting 8-(ii) from 8-(i), we have.

$$G(\vec{r}', \vec{r}'') - G(\vec{r}'', \vec{r}')$$
$$= -\int d^3r \; G(\vec{r}, \vec{r}'') \; \vec{\nabla} \cdot \left[\epsilon(\vec{r}) \; \vec{\nabla} G(\vec{r}, \vec{r}') \right]$$
$$+ \int d^3r \; G(\vec{r}, \vec{r}') \; \vec{\nabla} \cdot \left[\epsilon(\vec{r}) \; \vec{\nabla} G(\vec{r}, \vec{r}'') \right]$$

10 Integrating by parts we have

$$G(\vec{r}', \vec{r}'') - G(\vec{r}'', \vec{r}')$$
$$= -\int d^3r \; \vec{\nabla} \cdot \left[G(\vec{r}, \vec{r}'') \; \epsilon(\vec{r}) \; \vec{\nabla} G(\vec{r}, \vec{r}') \right]$$
$$+ \int d^3r \left[\vec{\nabla} G(\vec{r}, \vec{r}'') \right] \cdot \epsilon(\vec{r}) \left[\vec{\nabla} G(\vec{r}, \vec{r}') \right]$$
$$+ \int d^3r \; \vec{\nabla} \cdot \left[G(\vec{r}, \vec{r}') \; \epsilon(\vec{r}) \; \vec{\nabla} G(\vec{r}, \vec{r}'') \right]$$
$$- \int d^3r \left[\vec{\nabla} G(\vec{r}, \vec{r}') \right] \cdot \epsilon(\vec{r}) \left[\vec{\nabla} G(\vec{r}, \vec{r}'') \right]$$
$$= \int d^3r \; \vec{\nabla} \cdot \left[G(\vec{r}, \vec{r}') \; \epsilon(\vec{r}) \; \vec{\nabla} G(\vec{r}, \vec{r}'') - G(\vec{r}, \vec{r}'') \; \epsilon(\vec{r}) \; \vec{\nabla} G(\vec{r}, \vec{r}') \right]$$

11 Using divergence theorem in 10 we have.

$$G(\vec{r}', \vec{r}'') - G(\vec{r}'', \vec{r}')$$

$$= \oint_S d\vec{a} \cdot \left[G(\vec{r}, \vec{r}')\, \epsilon(\vec{r})\, \vec{\nabla} G(\vec{r}, \vec{r}'') - G(\vec{r}, \vec{r}'')\, \epsilon(\vec{r})\, \vec{\nabla} G(\vec{r}, \vec{r}') \right]$$

12 Let the surface S be that of a sphere. Integrating 7-(i) on the same sphere we obtain

$$-\oint_S d\vec{a} \cdot \left[\epsilon(\vec{r})\, \vec{\nabla} G(\vec{r}, \vec{r}') \right] = 1$$

13 Using 12 we can conclude

$\rightarrow \quad \epsilon \nabla G \sim \frac{1}{R^2}$ on the surface.

$\rightarrow$ Thus, $G \sim \frac{1}{R}$ (assuming $\epsilon = \epsilon_0$ on the surface).

$\rightarrow$ Thus, $G \epsilon \nabla G \sim \frac{1}{R^3}$.

$\rightarrow \quad \int d\vec{a} \cdot (G \epsilon \nabla G) \sim \frac{1}{R} \rightarrow 0$ for large R.

14 Thus, we have.

$$G(\vec{r}', \vec{r}'') = G(\vec{r}'', \vec{r}').$$

15 Next, let us state how the electric potential for an arbitrary charge distribution is given in terms of the Green's function. Let us begin by rewriting 4 and 5 here,

$$-\vec{\nabla}\cdot\left[\epsilon(\vec{r})\,\vec{\nabla}\,\phi(\vec{r})\right] = \rho(\vec{r}) \qquad \text{— (i)}$$

$$-\vec{\nabla}\cdot\left[\epsilon(\vec{r})\,\vec{\nabla}\,G(\vec{r},\vec{r}\,')\right] = \delta^{(3)}(\vec{r}-\vec{r}\,') \qquad \text{— (ii)}$$

16 Multiplying 15 -(i) by $G(\vec{r}\,',\vec{r})$, 15 -(ii) by $\phi(\vec{r})$, integrating over $\vec{r}$, we have

$$-\int d^3r\, G(\vec{r}\,',\vec{r})\,\vec{\nabla}\cdot\left[\epsilon(\vec{r})\,\vec{\nabla}\,\phi(\vec{r})\right] = \int d^3r\, G(\vec{r}\,',\vec{r})\,\rho(\vec{r}) \qquad \text{—(i)}$$

$$-\int d^3r\, \phi(\vec{r})\,\vec{\nabla}\cdot\left[\epsilon(\vec{r})\,\vec{\nabla}\,G(\vec{r},\vec{r}\,')\right] = \int d^3r\, \phi(\vec{r})\,\delta^{(3)}(\vec{r}-\vec{r}\,') \qquad \text{—(ii)}$$

17 Subtracting 16 -(i) from 16 -(ii) we have.

$$\phi(\vec{r}\,') - \int d^3r\, G(\vec{r}\,',\vec{r})\,\rho(\vec{r})$$

$$= -\int d^3r\, G(\vec{r}\,',\vec{r})\,\vec{\nabla}\cdot\left[\epsilon(\vec{r})\,\vec{\nabla}\,\phi(\vec{r})\right]$$

$$+\int d^3r\, \phi(\vec{r})\,\vec{\nabla}\cdot\left[\epsilon(\vec{r})\,\vec{\nabla}\,G(\vec{r},\vec{r}\,')\right]$$

(18) Integrating by parts we have.

$$\phi(\vec{r}') = \int d^3r \; G(\vec{r}', \vec{r}) \; \rho(\vec{r})$$
$$- \int d^3r \; \vec{\nabla} \cdot \left[G(\vec{r}', \vec{r}) \; \epsilon(\vec{r}) \; \vec{\nabla} \phi(\vec{r}) \right]$$
$$+ \int d^3r \left[\vec{\nabla} G(\vec{r}', \vec{r}) \right] \cdot \epsilon(\vec{r}) \left[\vec{\nabla} \phi(\vec{r}) \right]$$
$$+ \int d^3r \; \vec{\nabla} \cdot \left[\phi(\vec{r}) \; \epsilon(\vec{r}) \; \vec{\nabla} G(\vec{r}, \vec{r}') \right]$$
$$- \int d^3r \left[\vec{\nabla} \phi(\vec{r}) \right] \cdot \epsilon(\vec{r}) \left[\vec{\nabla} G(\vec{r}, \vec{r}') \right]$$

(19) The third and fifth term on right hand side cancel after using the reciprocity relation in (14), $G(\vec{r}, \vec{r}') = G(\vec{r}', \vec{r})$. Thus,

$$\phi(\vec{r}') = \int d^3r \; G(\vec{r}', \vec{r}) \; \rho(\vec{r})$$
$$+ \int d^3r \; \vec{\nabla} \cdot \left[\phi(\vec{r}) \; \epsilon(\vec{r}) \; \vec{\nabla} G(\vec{r}, \vec{r}') - G(\vec{r}', \vec{r}) \; \epsilon(\vec{r}) \; \vec{\nabla} \phi(\vec{r}) \right]$$

(20) Using divergence theorem we have

$$\phi(\vec{r}') = \int d^3r \; G(\vec{r}', \vec{r}) \; \rho(\vec{r})$$
$$+ \oint_S d\vec{a} \cdot \left[\phi(\vec{r}) \; \epsilon(\vec{r}) \; \vec{\nabla} G(\vec{r}, \vec{r}') - G(\vec{r}', \vec{r}) \; \epsilon(\vec{r}) \; \vec{\nabla} \phi(\vec{r}) \right].$$

(21) Rewriting (20) after replacing $\vec{r} \leftrightarrow \vec{r}'$ we have.

$$\phi(\vec{r}) = \int d^3r'\, G(\vec{r}, \vec{r}')\, \rho(\vec{r}') + \oint_S d\vec{a}' \cdot \left[\phi(\vec{r}')\, \epsilon(\vec{r}')\, \vec{\nabla}' G(\vec{r}', \vec{r}) - G(\vec{r}, \vec{r}')\, \epsilon(\vec{r}')\, \vec{\nabla}' \phi(\vec{r}') \right]$$

(22) Dirichlet boundary conditions $\left(\phi(\vec{r}')\big|_S = 0 \right)$

$$\phi(\vec{r}) = \int d^3r'\, G(\vec{r}, \vec{r}')\, \rho(\vec{r}') - \oint_S d\vec{a}' \cdot G(\vec{r}, \vec{r}')\, \epsilon(\vec{r}')\, \vec{\nabla}' \phi(\vec{r}')$$

(23) Neumann boundary conditions $\left(d\vec{a}' \cdot \vec{\nabla}' \phi(\vec{r}')\big|_S = 0 \right)$

$$\phi(\vec{r}) = \int d^3r'\, G(\vec{r}, \vec{r}')\, \rho(\vec{r}') + \oint_S d\vec{a}' \cdot \phi(\vec{r}')\, \epsilon(\vec{r}')\, \vec{\nabla}' G(\vec{r}', \vec{r})$$

(24) Robin boundary conditions are more general and are linear combinations of Dirichlet and Neumann boundary conditions,

$$a\, \phi(\vec{r}')\big|_S + b\, d\vec{a}' \cdot \vec{\nabla}' \phi(\vec{r}')\big|_S = 0,$$

where a and b are constants.