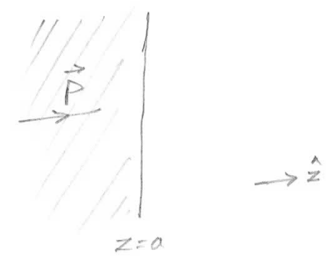
SET-14

Electrically polarized materials

(1) Example (1): A slab of infinite extent and infinite thickness occupying half of space. Let the polarizability of the slab be given by

$$\vec{P}(\vec{r},t) = \sigma\,\hat{z}\,\theta(a-z)$$

$$\theta(x) = \begin{cases} 1 & x>0 \\ 0 & x<0 \end{cases}$$



(2) Maxwell's equations for electrostatics, in the presence of materials are

$$\vec{\nabla}\cdot(\epsilon_0\vec{E}) = \underbrace{\rho}_{\text{free charge density}} - \underbrace{\vec{\nabla}\cdot\vec{P}}_{\text{effective charge density of a material}}$$

and

$$\vec{\nabla}\times\vec{E} = 0.$$

(3)
$$\begin{aligned}
\rho_{eff} &= -\vec{\nabla}\cdot\vec{P} \\
&= -\vec{\nabla}\cdot\left[\sigma\,\hat{z}\,\theta(a-z)\right] \\
&= -\sigma\,\hat{z}\cdot\vec{\nabla}\,\theta(a-z) \\
&= -\sigma\frac{\partial}{\partial z}\theta(a-z) \\
&= \sigma\,\delta(a-z)
\end{aligned}$$

$$\hat{z}\cdot\vec{\nabla} = \hat{z}\cdot\left[\hat{x}\frac{\partial}{\partial x}+\hat{y}\frac{\partial}{\partial y}+\hat{z}\frac{\partial}{\partial z}\right] = \frac{\partial}{\partial z}$$

physically interpret the effective charge density as an induced or permanent surface charge density.

4 $$\rho_{tot} = \underbrace{\rho}_{=0} - \vec{\nabla}\cdot\vec{P}$$

$$= \sigma\,\delta(a-z)$$

5 $$\phi(\vec{r}) = \frac{1}{4\pi\epsilon_0}\int d^3r' \frac{\rho_{tot}(\vec{r}')}{|\vec{r}-\vec{r}'|}$$

$$= \frac{1}{4\pi\epsilon_0}\int_{-\infty}^{+\infty}dx'\int_{-\infty}^{+\infty}dy'\int_{-\infty}^{+\infty}dz' \frac{\sigma\,\delta(a-z')}{\sqrt{(x-x')^2+(y-y')^2+(z-z')^2}}$$

$$= \frac{\sigma}{4\pi\epsilon_0}\int_{-\infty}^{+\infty}dx'\int_{-\infty}^{+\infty}dy' \frac{1}{\sqrt{(x-x')^2+(y-y')^2+(z-a)^2}}$$

$$x-x' = x'' \qquad y-y' = y''$$
$$-dx' = dx'' \qquad -dy' = dy''$$

$$= \frac{\sigma}{4\pi\epsilon_0}\int_{-\infty}^{+\infty}dx''\int_{-\infty}^{+\infty}dy'' \frac{1}{\sqrt{x''^2+y''^2+(z-a)^2}}$$

$$x''^2+y''^2 = R''^2 \qquad dx''dy'' = R''dR''d\theta''$$

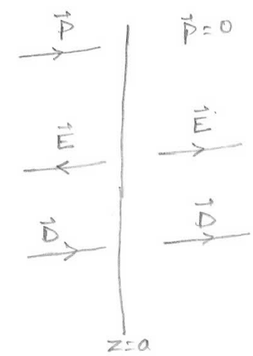
$$= \frac{\sigma}{4\pi\epsilon_0}\int_0^{2\pi}d\theta''\int_0^{\infty}R''dR'' \frac{1}{\sqrt{R''^2+(z-a)^2}}$$

$$= \frac{\sigma}{4\pi\epsilon_0}\,2\pi\int_0^{\infty}R''dR'' \frac{1}{\sqrt{R''^2+(z-a)^2}} \qquad \begin{aligned}R''^2+(z-a)^2 &= t''\\ 2R''dR'' &= dt''\end{aligned}$$

$$= \frac{\sigma}{2\epsilon_0}\int_{(z-a)^2}^{\infty\to R}\frac{dt''}{2}\frac{1}{\sqrt{t''}}$$

$$= \lim_{R\to\infty}\frac{\sigma}{2\epsilon_0}\Big[R - |z-a|\Big]$$ → a divergent expression, but, the physical quantity, the electric field comes out unambiguously finite.

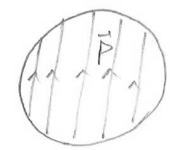
(6)
$$\vec{E} = -\vec{\nabla}\phi$$
$$= -\vec{\nabla}\left[\underset{R\to\infty}{\mathrm{Lt}} \frac{\sigma}{2\epsilon_0}\left(R - |z-a|\right)\right]$$
$$= -\underset{R\to\infty}{\mathrm{Lt}} \frac{\sigma}{2\epsilon_0} \vec{\nabla}\left(R - |z-a|\right)$$
$$= -\underset{R\to\infty}{\mathrm{Lt}} \frac{\sigma}{2\epsilon_0} \hat{z}\frac{\partial}{\partial z}\left(R - |z-a|\right)$$
(only z-component contributes.)
$$= \frac{\sigma}{2\epsilon_0} \hat{z}\frac{\partial}{\partial z}|z-a|$$
$$= \begin{cases} \dfrac{\sigma}{2\epsilon_0}\hat{z} & z > a \\ -\dfrac{\sigma}{2\epsilon_0}\hat{z} & z < a. \end{cases}$$



(7)
$$\vec{D} = \epsilon_0\vec{E} + \vec{P}$$
$$= \begin{cases} \dfrac{\sigma}{2}\hat{z} + 0 & z > a \\ -\dfrac{\sigma}{2}\hat{z} + \sigma\hat{z} & z < a. \end{cases}$$
$$= \frac{\sigma}{2}\hat{z}$$

(8) Comments. Note that for $\vec{P} = \sigma\,\hat{y}\,\theta(z-a)$ the effective charge density $\rho_{eff} = 0$.

(9) <u>Example (2)</u> : Consider a solid sphere of radius R with permanent polarization



$$\vec{P}(\vec{r},t) = \vec{P}_0 \, \theta(R-r).$$

$$\theta(x) = \begin{cases} 1 & x>0 \\ 0 & x<0 \end{cases}$$

(10) Maxwell's equations are.

$$\vec{\nabla}\cdot\left(\epsilon_0 \vec{E}\right) = \rho_{tot} = \rho \underbrace{- \vec{\nabla}\cdot\vec{P}}_{\rho_{eff}}$$

$$\vec{\nabla}\times\vec{E} = 0$$

(11)
$$\begin{aligned}
\rho_{eff} &= - \vec{\nabla}\cdot\vec{P} \\
&= - \vec{\nabla}\cdot\left[\vec{P}_0 \, \theta(R-r)\right] \\
&= - \vec{P}_0 \cdot \vec{\nabla}\, \theta(R-r) \\
&= - \vec{P}_0 \cdot (\vec{\nabla} r) \frac{\partial}{\partial r} \theta(R-r) \\
&= - \left(\vec{P}_0 \cdot \hat{r}\right) \frac{\partial}{\partial r} \theta(R-r) \\
&= \vec{P}_0 \cdot \hat{r} \, \delta(R-r)
\end{aligned}$$

(12)
$$\begin{aligned}
\rho_{tot} &= \underbrace{\rho}_{=0} - \vec{\nabla}\cdot\vec{P} \\
&= \vec{P}_0 \cdot \hat{r} \, \delta(R-r).
\end{aligned}$$

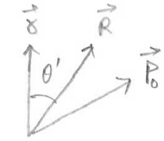
(13) The electric potential is determined using

$$\phi(\vec{r}) = \frac{1}{4\pi\epsilon_0} \int d^3r' \, \frac{\rho_{tot}(\vec{r}')}{|\vec{r} - \vec{r}'|}$$

$$= \frac{1}{4\pi\epsilon_0} \int_0^\infty r'^2 dr' \int_0^{2\pi} d\phi' \int_0^\pi \sin\theta' d\theta' \, \frac{(\vec{P}_0 \cdot \hat{r}') \, \delta(R - r')}{\sqrt{r^2 + r'^2 - 2\vec{r}\cdot\vec{r}'}}$$

$$= \frac{R^2}{4\pi\epsilon_0} \int_0^{2\pi} d\phi' \int_0^\pi \sin\theta' d\theta' \, \frac{\vec{P}_0 \cdot \{\hat{i} \sin\theta' \cos\phi' + \hat{j} \sin\theta' \sin\phi' + \hat{k} \cos\theta'\}}{\sqrt{r^2 + R^2 - 2\vec{r}\cdot\vec{R}}}$$

(14) Out of the three vectors $\vec{P}_0$, $\vec{r}$, and $\vec{R}$, choose $\vec{r}$ to be along the z-axis. Then



$$\vec{r} \cdot \vec{R} = r R \cos\theta'$$

(15) $$\phi(\vec{r}) = \frac{R^2}{4\pi\epsilon_0} \int_0^{2\pi} d\phi' \int_0^\pi \sin\theta' d\theta' \, \frac{\vec{P}_0 \cdot \{\hat{i} \sin\theta' \cos\phi' + \hat{j} \sin\theta' \sin\phi' + \hat{k} \cos\theta'\}}{\sqrt{r^2 + R^2 - 2rR\cos\theta'}}$$

Using $\int_0^{2\pi} d\phi' \cos\phi' = 0$, $\int_0^{2\pi} d\phi' \sin\phi' = 0$ we have.

$$\phi(\vec{r}) = \frac{(\vec{P}_0 \cdot \hat{k})}{4\pi\epsilon_0} \, 2\pi R^2 \int_0^\pi \sin\theta' d\theta' \, \frac{\cos\theta'}{\sqrt{r^2 + R^2 - 2Rr\cos\theta'}}$$

$\cos\theta' = t$
$-\sin\theta' d\theta' = dt$

$$= \frac{(\vec{P}_0 \cdot \hat{k})}{4\pi\epsilon_0} \, 2\pi R^2 \int_{-1}^{+1} dt \, \frac{t}{\sqrt{r^2 + R^2 - 2rRt}}$$

(16) Let

$$I(R,r) = \int_{-1}^{+1} dt \, \frac{t}{\sqrt{r^2+R^2-2rRt}} \qquad \begin{aligned} r^2+R^2-2rRt &= y \\ -2rR\,dt &= dy \end{aligned}$$

$$= \int_{(r+R)^2}^{(r-R)^2} -\frac{dy}{2rR}\,\frac{1}{\sqrt{y}}\,\frac{r^2+R^2-y}{2rR}$$

$$= \frac{1}{4r^2R^2}\int_{(r-R)^2}^{(r+R)^2} \frac{dy}{\sqrt{y}}\left[r^2+R^2-y\right]$$

$$= \frac{r^2+R^2}{4r^2R^2}\int_{(r-R)^2}^{(r+R)^2}\frac{dy}{\sqrt{y}} - \frac{1}{4r^2R^2}\int_{(r-R)^2}^{(r+R)^2} dy\,\sqrt{y}$$

$$= \frac{r^2+R^2}{4r^2R^2}\, 2\sqrt{y}\,\Big|_{(r-R)^2}^{(r+R)^2} - \frac{1}{4r^2R^2}\,\frac{2}{3}\,y^{\frac{3}{2}}\,\Big|_{(r-R)^2}^{(r+R)^2}$$

$$= \frac{r^2+R^2}{2r^2R^2}\left[(r+R)-|r-R|\right] - \frac{1}{6r^2R^2}\left[(r+R)^3-|r-R|^3\right]$$

(17) $r>R$:

$$I(R,r) = \frac{r^2+R^2}{2r^2R^2}\left[(r+R)-(r-R)\right] - \frac{1}{6r^2R^2}\left[(r+R)^3-(r-R)^3\right]$$

$$= \frac{r^2+R^2}{r^2R} - \frac{1}{6r^2R^2}\left[6r^2R+2R^3\right] = \frac{2}{3}\,\frac{R}{r^2}$$

$$= \frac{1}{6r^2R}\left[6r^2+6R^2-6r^2-2R^2\right]$$

$r<R$:

$$I(R,r) = \frac{r^2+R^2}{2r^2R^2}\left[(r+R)-(R-r)\right] - \frac{1}{6r^2R^2}\left[(r+R)^3-(R-r)^3\right]$$

$$= \frac{r^2+R^2}{rR^2} - \frac{1}{6r^2R^2}\left[6R^2r+2r^3\right] = \frac{2}{3}\,\frac{r}{R^2}$$

$$= \frac{1}{6rR^2}\left[6r^2+6R^2-6R^2-2r^2\right]$$

Thus,

$$\int_{-1}^{+1} dt\,\frac{t}{\sqrt{r^2+R^2-2rRt}} = \begin{cases} \dfrac{2}{3}\,\dfrac{r}{R^2} & r<R \\[2ex] \dfrac{2}{3}\,\dfrac{R}{r^2} & R<r \end{cases}$$

18 Using 17 in 15 we have.

$$\phi(\vec{r}) = \frac{(\vec{P}_0 \cdot \hat{k})}{4\pi\epsilon_0} \, 2\pi R^2 \begin{cases} \frac{2}{3}\frac{r}{R^2} & r < R \\ \frac{2}{3}\frac{R}{r^2} & R < r \end{cases}$$

$$= \frac{(\vec{P}_0 \cdot \hat{k})}{4\pi\epsilon_0} \begin{cases} \left(\frac{4\pi}{3} r^3\right) \frac{1}{r^2} & r < R \\ \left(\frac{4\pi}{3} R^3\right) \frac{1}{r^2} & R < r \end{cases}$$

Now, releasing the choice of $\vec{r}$ along $\hat{z}$ we have.

$$\phi(\vec{r}) = \frac{(\vec{P}_0 \cdot \hat{r})}{4\pi\epsilon_0} \begin{cases} \left(\frac{4\pi}{3} r^3\right) \frac{1}{r^2} & r < R \\ \left(\frac{4\pi}{3} R^3\right) \frac{1}{r^2} & R < r \end{cases}$$

19 The electric field is

$$\vec{E} = -\vec{\nabla}\phi$$

$$= -\frac{1}{4\pi\epsilon_0} \begin{cases} \frac{4\pi}{3} \vec{\nabla}\left[(\vec{P}_0 \cdot \hat{r})\, r\right] & r < R \\ \frac{4\pi}{3} R^3 \, \vec{\nabla}\left[\frac{(\vec{P}_0 \cdot \hat{r})}{r^2}\right] & R < r \end{cases}$$

(20) $$\vec{\nabla}\left[(\vec{P}_0\cdot\hat{r})\, r\right] = \vec{\nabla}(\vec{P}_0\cdot\vec{r}) = \vec{P}_0\cdot\vec{\nabla}\vec{r} = \vec{P}_0$$

dot

(21) $$\vec{\nabla}\left[\frac{(\vec{P}_0\cdot\hat{r})}{r^2}\right] = \vec{\nabla}\left[\frac{(\vec{P}_0\cdot\vec{r})}{r^3}\right]$$

$$= \left[\vec{\nabla}(\vec{P}_0\cdot\vec{r})\right]\frac{1}{r^3} + (\vec{P}_0\cdot\vec{r})\,\vec{\nabla}\frac{1}{r^3}$$

$$= \frac{\vec{P}_0}{r^3} - 3\frac{(\vec{P}_0\cdot\vec{r})}{r^4}\vec{\nabla} r$$

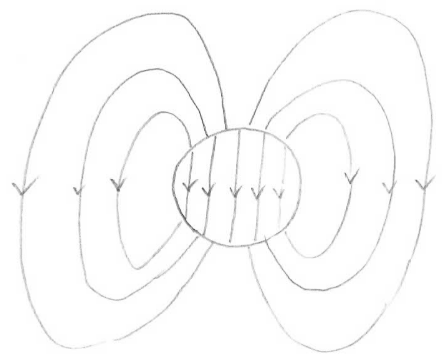
$$= \frac{1}{r^3}\left[\vec{P}_0 - 3(\vec{P}_0\cdot\hat{r})\,\hat{r}\right]$$

(22) Using (20) and (21) in (19) we have.

$$\vec{E} = \frac{1}{4\pi\epsilon_0}\begin{cases} -\frac{4\pi}{3}\vec{P}_0 & r < R \\ \left(\frac{4\pi}{3}R^3\right)\frac{1}{r^3}\left[3(\vec{P}_0\cdot\hat{r})\,\hat{r} - \vec{P}_0\right] & R < r \end{cases}$$

Thus, outside the sphere the electric field is that of a point dipole. Note, that it points opposite to $\vec{P}_0$ inside the sphere.

(23) Let us plot the electric field.



(24) $\vec{D} = \epsilon_0 \vec{E} + \vec{P}$

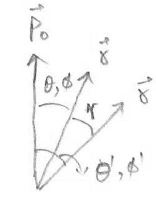
$$= \begin{cases} -\frac{1}{3}\vec{P}_0 + \vec{P}_0, & r < R \\ \frac{1}{4\pi}\left(\frac{4\pi}{3}R^3\right)\frac{1}{r^3}\left[3(\vec{P}_0\cdot\hat{r})\hat{r} - \vec{P}_0\right] & R < r \end{cases}$$

$$= \begin{cases} \frac{2}{3}\vec{P}_0 & r < R. \\ \frac{1}{4\pi}\left(\frac{4\pi}{3}R^3\right)\frac{1}{r^3}\left[3(\vec{P}_0\cdot\hat{r})\hat{r} - \vec{P}_0\right] & R < r \end{cases}$$

(25) What if we had chosen $\vec{P}_0$ to be along the z-axis, in (14)? The integrals become harder to evaluate. But, they can be performed in terms of Legendre polynomials, which are special cases of spherical harmonics. We will introduce these functions later in the course, but, they will used here to demonstrate the evaluation.

(26) Let $\vec{P}_0$ be along $\hat{z}$ direction in (13). Then,

$$\phi(\vec{r}) = \frac{R^2}{4\pi\epsilon_0} \int_0^{2\pi} d\phi' \int_0^{\pi} \sin\theta' d\theta' \frac{P_0 \cos\theta'}{\sqrt{r^2 + R^2 - 2rR\cos\gamma}}$$



where

$$\vec{r} = r(\sin\theta\cos\phi,\ \sin\theta\sin\phi,\ \cos\theta)$$

$$\vec{R} = R(\sin\theta'\cos\phi',\ \sin\theta'\sin\phi',\ \cos\theta')$$

$$\vec{r}\cdot\vec{R} = rR\cos\gamma = rR\left[\cos\theta\cos\theta' + \sin\theta\sin\theta'\cos(\phi-\phi')\right]$$

(27) In terms of Legendre polynomials, we have.

$$\cos\theta' = P_1(\cos\theta')$$

and

$$\frac{1}{\sqrt{r^2+R^2-2rR\cos\gamma}} = \sum_{l=0}^{\infty} \frac{r_<^l}{r_>^{l+1}} P_l(\cos\gamma)$$

$r_< = \mathrm{Min}(r, R)$

$r_> = \mathrm{Max}(r, R)$

$d\Omega' = \sin\theta' d\theta' d\phi'$

(28) Using (27) in (26)

$$\phi(\vec{r}) = \frac{P_0}{4\pi\epsilon_0} R^2 \int d\Omega' \, P_1(\cos\theta') \sum_{l=0}^{\infty} \frac{r_<^l}{r_>^{l+1}} P_l(\cos\gamma)$$

$$= \frac{P_0}{4\pi\epsilon_0} R^2 \sum_{l=0}^{\infty} \frac{r_<^l}{r_>^{l+1}} \int d\Omega' \, P_1(\cos\theta') \, P_l(\cos\gamma)$$

(29)

$$P_l(\cos\gamma) = \frac{4\pi}{2l+1} \sum_{m=-l}^{l} Y_{lm}(\theta,\phi) \, Y_{lm}(\theta',\phi')$$

$$P_1(\cos\theta') = \sqrt{\frac{4\pi}{3}} \, Y_{10}(\theta',\phi')$$

(30) Thus.

$$\phi(\vec{r}) = \frac{P_0}{4\pi\epsilon_0} R^2 \sum_{l=0}^{\infty} \frac{r_<^l}{r_>^{l+1}} \int d\Omega' \sqrt{\frac{4\pi}{3}} \, Y_{10}(\theta',\phi') \frac{4\pi}{2l+1} \sum_{m=-l}^{l} Y_{lm}(\theta,\phi) \, Y_{lm}(\theta',\phi')$$

$$= \frac{P_0}{4\pi\epsilon_0} R^2 \sum_{l=0}^{\infty} \sum_{m=-l}^{l} \frac{r_<^l}{r_>^{l+1}} \sqrt{\frac{4\pi}{3}} \frac{4\pi}{2l+1} Y_{lm}(\theta,\phi) \int d\Omega' \, Y_{10}(\theta',\phi') \, Y_{lm}(\theta',\phi')$$

31 Using orthogonality relation of spherical harmonics,

$$\int d\Omega' \, Y_{lm}(\theta',\phi') \, Y_{l'm'}(\theta',\phi') = \delta_{ll'} \, \delta_{mm'}$$

32 we have.

$$\phi(\vec{r}) = \frac{P_0}{4\pi\epsilon_0} R^2 \sum_{l=0}^{\infty} \sum_{m=-l}^{l} \frac{r_<^l}{r_>^{l+1}} \sqrt{\frac{4\pi}{3}} \frac{4\pi}{2l+1} Y_{lm}(\theta,\phi) \, \delta_{l1} \, \delta_{m0}$$

$$= \frac{P_0}{4\pi\epsilon_0} R^2 \frac{r_<}{r_>^2} \sqrt{\frac{4\pi}{3}} \frac{4\pi}{3} Y_{10}(\theta,\phi)$$

$$= \frac{P_0}{4\pi\epsilon_0} R^2 \frac{r_<}{r_>^2} \cancel{\sqrt{\frac{4\pi}{3}}} \frac{4\pi}{3} \cancel{\sqrt{\frac{3}{4\pi}}} \cos\theta \qquad \text{(using 29, 27)}$$

$$\vec{P}_0 \cdot \hat{k} = P_0 \cos\theta$$

$$= \frac{(\vec{P}_0 \cdot \hat{k})}{4\pi\epsilon_0} \frac{4\pi}{3} R^2 \frac{r_<}{r_>^2}$$

$$= \frac{\vec{P}_0 \cdot \hat{k}}{4\pi\epsilon_0} \begin{cases} \frac{4\pi}{3} r & r < R \\ \left(\frac{4\pi}{3} R^3\right) \frac{1}{r^2} & R < r. \end{cases}$$

which is exactly what we obtained in 18.